COMMENTS AND BIBLIOGRAPHY

for the course of Maria J. Esteban:

"Introduction to variational methods for non-compact problems"

In my course I treated three different situations :

- Minimization problems without compactness issues, in bounded domains and with subcritical nonlinearities. One of the problems involved a sublinear term and was solved by an unconstrained minimization method. Then, I adressed a superlinear, subcritical problem, for which I used a minimization method in a sphere (with an integral constraint).
- Then, we adressed the intermediate case of a minimization problem in the whole space $\mathbb{R}^N$, so in an unbounded domain, that we could still solve by "classical methods" using the symmetry properties of the domain and of the functional. We proved that the space of functions of $H^1(\mathbb{R}^N)$ which are radially symmetric embeds itself compactly in $L^q(\mathbb{R}^N)$ for $2 < q < 2N/(N-2),\ N \geq 3$. Using this compactness property, we could solve the minimization problem by the same method used to solve "compact" problems.
- Finally we adressed the case of minimization problems in the whole space and without symmetry. In order to analyze the behavior of minimizing sequences and the existence or nonexistence of minimizers I stated and proved the first concentration-compactness lemma, due to P.-L. Lions (see [3]). This lemma is a very powerful tool which allows to analyze the possible compactness defects of bounded sequences in Sobolev spaces. We finished the course by treating a superlinear subcritical elliptic equation in $\mathbb{R}^N$, without symmetry. We used the concentration-compactness method to analyze the behavior of all minimizing sequences of the corresponding minimization problem. This allowed us to select cases in which all minimizing sequences were relatively compact and therefore, we could find a minimum. Cases in which there was a minimum despite the fact that some minimizing sequences were not relatively compact. And cases in which no minimizing sequence was compact, and therefore, there was no minimum.

Note that there was no time to analyze losses of compactness due to concentration phenomena. All we did was related to possible losses of compactness due to translation invariance.

In order to get more information about these kinds of techniques, I add to these notes a short bibliography with articles and books which deal with the problem, are clear and pedagogic and contain many examples.

Articles [2] and [7] are related to compactness via symmetry and also to compact problems. Kavian's book is a very good presentation of variational methods in general at a not too high level of difficulty. Ghoussoub's and Struwe's books are more elaborate and contain more complicated examples, so they are more difficult to follow. Willem's book is also very accessible.

References

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[3] P.-L. Lions. The concentration-compactness method in the Calculus of Variations. The locally compact case. Part. I: Anal. non-linéaire, Ann. IHP **1** (1984), p. 109-145. Part. II: Anal. non-linéaire, Ann. IHP **1** (1984), p. 223-283.

[4] P.-L. Lions. Solutions of Hartree-Fock equations for Coulomb systems. Comm. Math. Phys. **109** (1987), p. 33-97.

[5] G. Fang, N. Ghoussoub. Morse-type information on Palais-Smale sequences obtained by min-max principles. Manuscripta Math. **75** (1992), p. 81-95.

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[7] Strauss, Walter A. Existence of solitary waves in higher dimensions. Comm. Math. Phys. 55 (1977), no. 2, 149–162.

[8] Struwe, Michael. *Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems.* Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 34. Springer-Verlag, Berlin, 1996.

[9] Willem, Michel *Minimax theorems.* Progress in Nonlinear Differential Equations and their Applications, 24. Birkhuser Boston, Inc., Boston, MA, 1996. x+162 pp. ISBN: 0-8176-3913-6