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Identification of Robin coefficients by the means of boundary measurements

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Abstract. We consider the problem of determining the Robin coefficient of some specimen material, by performing measurements on some part of the boundary. An identifiability result is proved for Robin coefficients which are continuous functions with some negative lower bound. Both local and monotone global Lipschitz stability results are established. Finally, a cost function turning the inverse problem into an optimization one is proposed for numerical purposes. This function, which may be viewed as an energetic least-squares one, has an easy-to-compute G-derivative, which encourages us to consider implementing the gradient algorithm in forthcoming numerical experiments.

1. Introduction

In this paper we are interested in determining the Robin coefficient φ of some material of which a body occupying the connected domain Ω in $\mathbb{R}^2$ or $\mathbb{R}^3$ is composed. To this end we shall use boundary measurements of the temperature on some part K of the boundary $\partial\Omega$, which is assumed to be of $\mathcal{C}^{1,1}$ regularity, and moreover, let γ, Γ_D and Γ_N be three open subsets of the boundary such that

$$\partial\Omega = \overline{\gamma} \cup \overline{\Gamma}_D \cup \overline{\Gamma}_N.$$

The direct problem associated with the inverse one we deal with is therefore the following:

$$\begin{aligned} &\Delta u = 0 && \text{in} \quad \Omega \\ &u = 0 && \text{on} \quad \Gamma_D \\ &\frac{\partial u}{\partial n} = \phi && \text{on} \quad \Gamma_N \\ &\frac{\partial u}{\partial n} + \varphi u = 0 && \text{on} \quad \gamma. \end{aligned} \tag{1}$$

We use the language of thermal imaging, ϕ thus being the prescribed heat flux on Γ_N ($\phi \not\equiv 0$ on Γ_N), and φ the unknown heat-exchange function, which has to be determined by measuring the temperature on some open subset K of Γ_N; $f = u|_K$. However, problem (1) might also be viewed as a model for a corrosion detection problem by voltage measurements (see, for example, [15, 16]): φ is thus the corrosion coefficient, ϕ the current flux prescribed on Γ_N and u the electrostatic potential.

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Given the flux ϕ and the measured temperature f, the inverse problem is, after defining an appropriate set Φ_{ad} of admissible heat-exchange coefficients, the following:

$$\text{Find } \varphi \in \Phi_{ad} \text{ such that: } u, \text{ the solution of (1), also verifies } u|_K = f. \qquad (2)$$

The results reported in the present paper are related to the following three issues.

Identifiability. Studying the case, very close to ours, of corrosion detection, Inglese [15] proved that in a 2D situation a single measurement of u on K suffices to uniquely determine the coefficient φ, for φ ranging over the set

$$\Phi_{ad} = \{\varphi \in \mathcal{C}_0^3(\overline{\gamma}) \text{ such that } \varphi(x) \geqslant 0 \quad \forall x \in \overline{\gamma} \text{ and } \varphi \not\equiv 0\}$$

K being any non-empty open subset of Γ_N. Our contribution in this paper has been to expand the set of admissible coefficients to continuous functions on $\bar{\gamma}$, and to give a proof of the uniqueness result valid in 2D as well as in 3D situations. More recently, Choulli [9] proved an identifiability result in the case of a nonlinear heat law.

Stability. This means the continuous dependence of the unknown parameter on the measured data, which is a crucial issue for numerical applications, and has been the concern of many authors. Among them, Alessandrini [1] and Bellout and Friedman [5] dealt with the stability for an inverse conductivity problem, and gave a definition for the local Lipschitz stability which has been repeatedly used by several authors. For cracks and boundary recovery, one may cite Friedman and Vogelius [11], Andrieux *et al* [3], Ben Abda *et al* [6, 7], and a large number of results presented in papers by Alessandrini *et al* since 1988, e.g. [1], these results being somewhat summarized in the paper by Alessandrini and DiBenedetto [2]. In his previously mentioned paper, Inglese [15] proved the continuity, and the Gateaux differentiability of the direct map which associates with a given conductivity, φ, the measured data on the boundary. In the present work, we first prove a local Lipschitz stability result, derived from the Gateaux differentiability, by establishing that the G-derivative does not vanish. By thoroughly studying the behaviour of the heat field u with respect to the Robin coefficient φ, we are also able to derive a more global Lipschitz stability result.

Identification. Most of the numerical algorithms worked out for identification purposes are based on a least-squares approach. Following Kohn and Vogelius [18], here we propose a 'variational least-squares' cost function representing, for a given Robin coefficient φ, the energy gap between the 'Neumann' field computed using the prescribed flux ϕ as a boundary condition, and the 'Dirichlet' one corresponding to the measured data f as a boundary condition. It turns out that the unique minimum of this cost function is the unknown coefficient, and that its expression splits as the sum of two uncoupled compliance functions, which makes its gradient easy to compute. This idea, which was first used for numerical implementations to identify conductivities by Kohn and McKenney [17], generalizes quite well to several other situations (for examples, see Andrieux *et al* [3], Roche and Sokolowski [21], Chaabane and Jaoua [8], etc). A recovery algorithm based on the gradient method is therefore proposed. Thanks to the cost function features reported above, its G-derivative with respect to φ in any direction ψ can be explicitly computed, and its expression depends only on the state u^0, not on its derivative $u^1(\psi)$.

2. Identifiability

Let V be the following space:

$$V = \{u \in H^1(\Omega) \text{ such that } u|_{\Gamma_D} = 0\} \tag{3}$$

V is a Hilbert space with respect to the inner product defined by $(u, v) := \int_\Omega \langle \nabla u, \nabla v\rangle$. Denoting by $|.|_{1,\Omega}$ the norm derived from this inner product, let α be the norm of the trace operator

$$\begin{aligned} \tau : V &\longrightarrow H^{\frac{1}{2}}(\partial\Omega) \\ u &\longmapsto u|_{\partial\Omega} \end{aligned} \tag{4}$$

while considered as a mapping from V equipped with the energy norm, onto $L^2(\partial\Omega)$:

$$\alpha = \sup_{v\in V;\ v\neq 0} \frac{|v|_{0,\partial\Omega}}{|v|_{1,\Omega}}.$$

The set of admissible Robin coefficients is defined as follows:

$$\Phi_{ad} = \left\{\varphi \in \mathcal{C}^0(\bar{\gamma});\ \min_{x\in\bar{\gamma}} \varphi(x) > -\frac{1}{\alpha^2}\right\}. \tag{5}$$

The following identifiability result presented in theorem 1 then holds.

Theorem 1 (uniqueness). *Let φ_1 and φ_2 be two elements of Φ_{ad}, and $(u_i)_{i=1,2}$ be the solutions of problem (1) with $(\varphi_i)_{i=1,2}$ as a Robin coefficient. Suppose that $u_1|_K = u_2|_K$. Then, $\varphi_1 = \varphi_2$.*

Proof. Let φ_1 and φ_2 be two elements of Φ_{ad} such that: $u_1|_K = u_2|_K$, and let us denote by w their difference ($w = u_1 - u_2$), which is a solution of the following problem:

$$\begin{aligned} &\Delta w = 0 && \text{in} \quad \Omega \\ &w = 0 && \text{on} \quad K \\ &\frac{\partial w}{\partial n} = 0 && \text{on} \quad \Gamma_N. \end{aligned} \tag{6}$$

By using Holmgren's unique continuation theorem, we get $w \equiv 0$ in Ω, which means that $u_1 = u_2$ in Ω and therefore:

$$\begin{aligned} &\frac{\partial u_1}{\partial n} + \varphi_1 u_1 = 0 && \text{on} \quad \gamma \\ &\frac{\partial u_1}{\partial n} + \varphi_2 u_1 = 0 && \text{on} \quad \gamma. \end{aligned} \tag{7}$$

Thus

$$u_1(\varphi_1 - \varphi_2) = 0 \qquad \text{on} \quad \gamma. \tag{8}$$

Let us assume that $\varphi_1 \not\equiv \varphi_2$. Thanks to the continuity of φ_1 and φ_2, we can find some open subset ϑ of γ with positive measure such that

$$(\varphi_1 - \varphi_2)(x) \neq 0 \qquad \forall x \in \vartheta.$$

Equation (8) then yields $u_1 \equiv 0$ on ϑ, and u_1 is therefore a solution of the Cauchy problem

$$\begin{aligned} &\Delta u_1 = 0 && \text{in} \quad \Omega \\ &u_1 = 0 && \text{on} \quad \vartheta \\ &\frac{\partial u_1}{\partial n} = 0 && \text{on} \quad \vartheta. \end{aligned} \tag{9}$$

If we use Holmgren's theorem again, we get $u_1 \equiv 0$ in Ω, which is in contradiction with $\phi \not\equiv 0$ on Γ_N. $\square$

Remark. Dropping the Dirichlet boundary condition ($\Gamma_D = \emptyset$), the same result holds with the space of admissible Robin coefficients below:

$$\Phi_{ad} = \{\varphi \in \mathcal{C}^0(\bar{\gamma});\ \varphi(x) \geqslant 0;\ \varphi(x) \not\equiv 0\} \tag{10}$$

the non-negativity of φ being necessary to ensure that the direct problem is well posed (see, for example, Garabedian [12]).

3. Stability

The measurements are assumed to be performed on a non-empty open subset K of Γ_N. Roughly speaking, stability means that small errors in the mesurements would yield small perturbations on the unknown coefficient φ. To formalize the idea, let us consider the mapping η defined by

$$\begin{aligned}\eta : \Phi_{ad} &\longmapsto L^2(K)\\ \varphi &\longmapsto f = u_\varphi|_K.\end{aligned}$$

The identifiability result proved above means that η is injective, and therefore, that the restriction

$$\eta : \Phi_{ad} \longrightarrow \eta(\Phi_{ad})$$

is invertible. Limiting our search to any compact subset of Φ_{ad}, a weak stability result, which is merely the continuity of the inverse operator η^{-1}, can be obtained as a straightforward consequence of the uniqueness theorem (Andrieux *et al* [4]). For numerical purposes however, such a result is scarcely sufficient. This is the reason why we shall be focusing our attention on Lipschitz stability, even if the results proved in this domain might not be as global as one would wish them to be.

3.1. Local Lipschitz stability

To prove local Lipschitz stability, the Lagrangian differentiation with respect to the domain has been repeatedly used as a basic and somewhat powerful tool for the study of geometric inverse problems ([3, 4, 6], etc). The Robin boundary condition has also been studied in [10], in the framework of 2D cracks recovery, a local Lipschitz stability result being proved using the same technique. Actually, the latter also works for differentiation with respect to the field or boundary coefficients, as was stated many years ago by Simon [22].

Now given $\varphi \in \Phi_{ad}$, and $\psi \in \Phi_{ad}$, there exists some real number $h_0 > 0$, depending on φ and ψ, such that

$$h \in]-h_0; h_0[\Rightarrow \varphi^h := (\varphi + h\psi) \in \Phi_{ad}.$$

Let u^h be the solution of problem (1) for φ^h as a Robin coefficient. We then have the following proposition.

Proposition 1. *There exist u^1 and $\varepsilon(h)$ in $H^1(\Omega)$ such that*

$$u^h = u^0 + hu^1 + h\varepsilon(h) \tag{11}$$

where $\lim_{h\to 0}|\varepsilon(h)|_{1,\Omega}=0$*,* u^0 *is solution of (1) with* φ *as a Robin coefficient, and* u^1 *is solution of the following problem:*

Find u^1 *in* V *such that*

$$\int_\Omega \langle\nabla u^1,\nabla v\rangle+\int_\gamma \varphi u^1 v=-\int_\gamma \psi u^0 v \qquad \textit{for all } v\in V. \tag{12}$$

Proof. The variational formulation of problem (1) is:

Find u in V such that

$$\int_\Omega \langle\nabla u,\nabla v\rangle+\int_\gamma \varphi u v=\int_{\Gamma_N}\phi v \qquad \text{for all } v\in V. \tag{13}$$

Let V^* be the dual space of V, and define the following mapping:

$$\begin{aligned}&\Lambda:\,]-h_0,h_0[\,\times V\longmapsto V^*\\ &(h,u)\longmapsto\left\{v\mapsto\int_\Omega\langle\nabla u,\nabla v\rangle+\int_\gamma\varphi^h uv-\int_{\Gamma_N}\phi v\right\}.\end{aligned}$$

The solution u^h of problem (1) with φ^h as a Robin coefficient is therefore the solution of $\Lambda(h,u^h)=0$. The mapping Λ being linear with respect to u, its partial derivative with respect to u ($\frac{\partial\Lambda}{\partial u}(0,u)=\Lambda(0,\cdot)$) turns out to be an isomorphism between V and V^*. By the implicit function theorem, it turns out that u^h is some $\mathcal{C}^1$ function of h, for h small enough. This yields expansion (11), with $u^1\in V$, and $\lim_{h\to 0}|\varepsilon(h)|_{1,\Omega}=0$.

Now, plugging this expansion in the variational formulation (13) of the direct problem with $\varphi^h=\varphi+h\psi$ as a Robin coefficient, and identifying in both sides of the equation terms of the same order in h, we derive that:

- u^0 is a solution of problem (1) with φ as a Robin coefficient
- u^1 is a solution of problem (12)

which ends the proof. □

Theorem 2 (local Lipschitz stability). *Assume that* $\psi\not\equiv 0$ *on* γ*, and denote by* $f^h=u^h|_K$*. Then*

$$\lim_{h\to 0}\frac{|f^h-f^0|_{0,K}}{|h|}>0. \tag{14}$$

Proof. According to expansion (11), (14) is equivalent to

$$|u^1|_{0,K}>0. \tag{15}$$

Let us then assume that $u^1=0$ on K. In this case, equation (12) gives that u^1 is a solution of the Cauchy problem

$$\begin{aligned}\Delta u^1&=0 &&\text{in}\quad\Omega\\ u^1&=0&&\text{on}\quad K\\ \frac{\partial u^1}{\partial n}&=0&&\text{on}\quad\Gamma_N\end{aligned} \tag{16}$$

which by Holmgren's theorem leads to

$$u^1\equiv 0 \qquad \text{in}\quad\Omega.$$

Equation (12) then yields

$$\int_\gamma \psi u^0 v=0 \qquad \forall v\in V$$

from which we derive

$$\psi u^0 = 0 \qquad \text{a.e. in } \gamma.$$

From the continuity of ψ, and the fact that $\psi \not\equiv 0$ on γ, we derive the existence of some open subset ϑ of γ where

$$u^0 = 0 \qquad \text{on} \quad \vartheta.$$

The Robin boundary condition yields

$$\frac{\partial u^0}{\partial n} = 0 \qquad \text{on} \quad \vartheta.$$

Using Holmgren's theorem again, we derive that $u^0 \equiv 0$ in Ω, which is in contradiction with our assumption that $\phi \not\equiv 0$ on Γ_N. ☐

3.2. *Behaviour of the solution u_φ with respect to φ*

In the following, we shall assume that:

- The flux ϕ is some non-negative function of $L^p(\Gamma_N)$ $(p > 2)$ such that $\phi \not\equiv 0$ on Γ_N.
- The class of admissible coefficients φ is restricted to

$$\Phi'_{ad} = \{\varphi \in \mathcal{C}^{0,1}(\bar{\gamma}) \text{ such that } \varphi(x) \geqslant 0\ \forall x \in \gamma\}.$$

None of these assumptions should actually be viewed as a limitation: the second one is in agreement with physics, and the first one is related to the prescribed flux used for the measurements, which is under control and may therefore be shaped in order to meet the needs of the identification process.

For $\varphi \in \Phi'_{ad}$, let us denote by u_φ the solution of problem (1) with φ as a Robin coefficient, and let us consider the following map:

$$\begin{aligned} \mathcal{U} : \Phi'_{ad} &\longmapsto V \\ \varphi &\longmapsto u_\varphi. \end{aligned} \tag{17}$$

The behaviour of u_φ with respect to φ is characterized by positivity and monotonicity, as stated in lemmas 2 and 3 which follow. First, let us recall a regularity result due to Grisvard [14], which we shall be using throughout this section.

Lemma 1 (regularity, Grisvard). *Let $\phi \in L^p(\Gamma_N)$ for some $p > 2$, and $\varphi \in \Phi_{ad}$. Therefore, the solution u_φ of problem (1), with ϕ as a prescribed flux and φ as a Robin coefficient, is continuous on $\overline{\Omega}$.*

This result is, as pointed out for 2D polygonal domains by Mghazli [19], a consequence of the mixed Dirichlet–Neumann boundary value problem solutions regularity. Indeed, the variational formulation of problem (1) is in $H^1(\Omega)$, and the trace of its solution $u_\varphi|_\gamma$ is therefore in $H^{\frac{1}{2}}(\gamma)$. Sobolev's imbeddings yield $H^{\frac{1}{2}}(\gamma) \subset L^q(\gamma)$ for any $q < 4$ in three dimensions, and for any $q < \infty$ in two dimensions (see, for example, [13]). Considering u_φ as the solution of the following mixed boundary value problem:

$$\begin{aligned} &\Delta u_\varphi = 0 && \text{in} \quad \Omega \\ &u_\varphi = 0 && \text{on} \quad \Gamma_D \\ &\frac{\partial u_\varphi}{\partial n} = \phi && \text{on} \quad \Gamma_N \\ &\frac{\partial u_\varphi}{\partial n} = -\varphi u_\varphi && \text{on} \quad \gamma \end{aligned} \tag{18}$$

where the prescribed Neumann boundary data are therefore an $L^r(\gamma \cup \Gamma_N)$ function, where r is defined by

$$r = \begin{cases} p & \text{if } p < 4 \\ \text{any } r;\ 2 < r < 4 & \text{if } p \geqslant 4. \end{cases}$$

Addressing such issues, Grisvard [14] proved that the solution of such a problem splits into a regular part u^R, the smoothness of which is consistent with the data according to the shift theorem (in this case $u^R \in W^{1+\frac{1}{r},r}(\Omega)$), and a 'singular part' $\mathcal{S}$ which is some explicitly known function of the square root of the distance to the boundary separating the Dirichlet part from the Neumann part of Γ. As a result of Sobolev's imbeddings, this singular part is continuous on $\bar{\Omega}$, as is the smooth part.

Remarks.

(1) To hold, this regularity result does not require that φ belongs to Φ_{ad}, but only to $L^\infty(\gamma)$.
(2) In the case $\Gamma_D = \emptyset$, the regularity is merely a matter of the shift theorem for the Neumann problem.

Lemma 2 (positivity). *For any function φ of Φ'_{ad}, we have*

$$u_\varphi(x) \geqslant 0 \qquad \forall x \in \bar{\Omega}.$$

Furthermore, let κ be any compact subset of γ. Then

$$\min_{x\in\kappa} u_\varphi(x) > 0.$$

Proof. The positivity of u_φ is a classical result (see, for example, Protter and Weinberger [20]). Suppose now, that for some compact subset κ of γ, we have

$$\min_{x\in\kappa} u_\varphi(x) = 0.$$

There then exists some $x_0 \in \kappa$ such that $u_\varphi(x_0) = 0$. x_0 is therefore a global minimum for u_φ, and the Robin boundary condition gives that $\frac{\partial u_\varphi}{\partial n}(x_0) = 0$. Thanks to lemma 1, Hopf's maximum principle (see, for example, Gilbarg and Trudinger [13]) applies, from which we derive that $u_\varphi \equiv 0$ in Ω, which is in contradiction with $\phi \not\equiv 0$ on Γ_N. □

Lemma 3 (monotonicity). *Let φ_1 and φ_2 be two elements of Φ'_{ad} such that*

$$\varphi_1(x) \geqslant \varphi_2(x) \qquad \forall x \in \overline{\gamma}$$

and let $u_i = \mathcal{U}(\varphi_i)$; $i = 1, 2$. Therefore:

$$u_1(x) \leqslant u_2(x) \qquad \forall x \in \bar{\Omega}.$$

Proof. Let $w = u_1 - u_2$, which therefore solves the following:

$$\begin{aligned} &\Delta w = 0 && \text{in } \Omega \\ &w = 0 && \text{on } \Gamma_D \\ &\frac{\partial w}{\partial n} = 0 && \text{on } \Gamma_N \\ &\frac{\partial w}{\partial n} + \varphi_1 w = -(\varphi_1 - \varphi_2)u_2 && \text{on } \gamma. \end{aligned} \tag{19}$$

Let $\mathcal{M}$ be the maximum of w, and suppose $\mathcal{M} > 0$. Therefore, this maximum is not achieved on $\overline{\Gamma}_D$. Let $x_1 \in \partial\Omega \setminus \overline{\Gamma}_D$ such that $w(x_1) = \mathcal{M}$. Since w is not constant on $\bar{\Omega}$, by Hopf's maximum principle we get

$$\frac{\partial w}{\partial n}(x_1) > 0.$$

It follows that $x_1 \notin \Gamma_N$, and therefore $x_1 \in \bar{\gamma}$. Two cases have to be considered at this point.

First case: $x_1 \notin \partial\gamma$. Using the Robin boundary condition, we get

$$\frac{\partial w}{\partial n}(x_1) = -\varphi_1(x_1)\mathcal{M} + (\varphi_2 - \varphi_1)(x_1)u_2(x_1)$$

which by using the positivity lemma 2 leads to $\frac{\partial w}{\partial n}(x_1) \leqslant 0$, which is contradictory with the above outcome of Hopf's maximum principle.

Second case: $x_1 \in \partial\gamma$. In this case, the Robin boundary condition does not hold at point x_1 since $\frac{\partial w}{\partial n}$ lacks continuity. Let us choose some real number p, $2 < p < 4$, and let q be defined by $q = \frac{2p}{p-2}$ (thus $\frac{p}{2}$ and $\frac{q}{2}$ are conjugate numbers). Therefore, let $(\alpha_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ be two sequences of non-negative $\mathcal{C}_0^\infty(\gamma)$ functions such that:

- $\alpha_n \longrightarrow \varphi_1$ in $L^q(\gamma)$
- $g_n \longrightarrow (\varphi_1 - \varphi_2)u_2$ in $L^2(\gamma)$.

Then let w_n be the solution of

$$\begin{aligned} &\Delta w_n = 0 && \text{in} \quad \Omega \\ &w_n = 0 && \text{on} \quad \Gamma_D \\ &\frac{\partial w_n}{\partial n} = 0 && \text{on} \quad \Gamma_N \\ &\frac{\partial w_n}{\partial n} + \alpha_n w_n = -g_n && \text{on} \quad \gamma \end{aligned} \tag{20}$$

and let $\tilde{\alpha}_n$ and $\tilde{g}_n$ be the functions defined on $\partial\Omega \setminus \overline{\Gamma}_D$, by extending α_n and g_n by zero outside γ. Therefore, w_n verifies

$$\frac{\partial w_n}{\partial n} + \tilde{\alpha}_n w_n = -\tilde{g}_n \qquad \text{on} \quad \partial\Omega \setminus \overline{\Gamma}_D.$$

Let us first prove that $w_n \leqslant 0$. To this end, we define $\mathcal{M}_n = \max_{x\in\overline{\Omega}} w_n$. This maximum is achieved on some point x_n of the boundary. If $x_n \in \overline{\Gamma}_D$, then $\mathcal{M}_n = 0$ and $w_n \leqslant 0$ on $\bar{\Omega}$ which ends the case.

Let us then suppose that $\mathcal{M}_n > 0$ and thus that $x_n \notin \overline{\Gamma}_D$. By Hopf's maximum principle:

- either w_n is constant, and thus $w_n \equiv 0$ in $\overline{\Omega}$
- or $\frac{\partial w_n}{\partial n}(x_n) > 0$.

However, the second occurence above is not possible since by expressing the Robin boundary condition at the point x_n, and using the non-negativity of $\tilde{g}_n$, $\tilde{\alpha}_n$ and $\mathcal{M}_n$, we get $\frac{\partial w_n}{\partial n}(x_n) \leqslant 0$.

It follows that $w_n \equiv 0$ in Ω, which is in contradiction with $\mathcal{M}_n > 0$. Hence, $w_n \leqslant 0$ in $\overline{\Omega}$.

Now, let $(\Theta_i)_{i=1,2}$ solve the following problems, with $\zeta_i \in \Phi_{ad}$ and $\chi_i \in L^2(\gamma)$; $i = 1, 2$:

$$\begin{array}{lll} \Delta\Theta_i = 0 & \text{in} & \Omega \\ \Theta_i = 0 & \text{on} & \Gamma_D \\ \dfrac{\partial\Theta_i}{\partial n} = \phi & \text{on} & \Gamma_N \\ \dfrac{\partial\Theta_i}{\partial n} + \zeta_i\Theta_i = \chi_i & \text{on} & \gamma. \end{array} \tag{21}$$

Given the real number q defined above, the following *a priori* estimate comes out by using the imbedding of $H^{\frac{1}{2}}(\gamma)$ into $L^p(\gamma)$, which holds in either the 2D or 3D cases, and the Cauchy–Schwarz inequality as well as the Hölder inequality for the two conjugate numbers $\frac{p}{2}$ and $\frac{q}{2}$:

$$|\Theta_1 - \Theta_2|_{1,\Omega} \leqslant \alpha(|\chi_1 - \chi_2|_{0,\gamma} + |\zeta_1 - \zeta_2|_{0,q,\gamma}|\Theta_2|_{0,p,\gamma}).$$

Applied to w and w_n, which both solve problems of type (21), this estimate yields that $w_n \to w$ in $H^1(\Omega)$, and therefore that $w \leqslant 0$ since w_n is, so that $\mathcal{M} = 0$. This ends the proof. □

3.3. *A monotone Lipschitz stability result*

Gathering the results obtained in the previous section, we are now able to prove a global monotone Lipschitz stability result. The estimate we get holds under two restrictions:

- φ and ψ have to be bounded, and comparable.
- The Lipschitz dependence is not achieved on the whole set γ, but only on any compact subset κ of γ.

Theorem 3 (monotone Lipschitz stability). *Let κ be a connected compact subset of γ and M a positive constant. Then, there exists some constant $c' = c'(\kappa, M) > 0$ such that given any pair $(\varphi, \psi) \in (\Phi'_{ad})^2$ verifying $\varphi \leqslant \psi \leqslant M$, we have*

$$|\varphi - \psi|_{0,1,\kappa} \leqslant c'|f_\varphi - f_\psi|_{0,1,K}. \tag{22}$$

Proof. Let u_M be the solution of problem (1) with the Robin coefficient with constant value M. By lemmas 2 and 3, we get

$$\min_{x\in\kappa} u_\psi(x) \geqslant \min_{x\in\kappa} u_M(x) > 0 \tag{23}$$

and thus

$$\min_{x\in\kappa} u_\psi(x) \geqslant m_\kappa := \min_{x\in\kappa} u_M(x).$$

Let $w = u_\varphi - u_\psi$, and z_φ be the solution of the problem

$$\begin{array}{lll} \Delta z_\varphi = 0 & \text{in} & \Omega \\ z_\varphi = 0 & \text{on} & \Gamma_D \\ \dfrac{\partial z_\varphi}{\partial n} = 1 & \text{on} & K \\ \dfrac{\partial z_\varphi}{\partial n} = 0 & \text{on} & \Gamma_N \setminus K \\ \dfrac{\partial z_\varphi}{\partial n} + \varphi z_\varphi = 0 & \text{on} & \gamma \end{array} \tag{24}$$

where w is the solution of the following variational problem:

$$\begin{aligned}&\text{Find } w \text{ in } V \text{ such that}\\ &\int_\Omega \langle \nabla w, \nabla v\rangle + \int_\gamma \varphi w v + \int_\gamma (\varphi - \psi) u_\psi v = 0 \qquad \text{for all } v \in V.\end{aligned} \tag{25}$$

Choosing z_φ as a test function, we get

$$\int_\Omega \langle \nabla w, \nabla z_\varphi\rangle + \int_\gamma \varphi w z_\varphi + \int_\gamma (\varphi - \psi) u_\psi z_\varphi = 0.$$

Integrating by parts, we obtain

$$\int_K w = \int_\gamma (\psi - \varphi) u_\psi z_\varphi.$$

Thanks to the positivity of u_ψ, z_φ and $\psi - \varphi$, this yields

$$\int_K w = \int_\gamma (\psi - \varphi) u_\psi z_\varphi \geqslant \int_\kappa (\psi - \varphi) u_\psi z_\varphi. \tag{26}$$

Now let ζ_M be the solution of the following problem:

$$\begin{array}{lll} \Delta\zeta_M = 0 & \text{in} & \Omega \\ \zeta_M = 0 & \text{on} & \Gamma_D \\ \dfrac{\partial \zeta_M}{\partial n} = 1 & \text{on} & K \\ \dfrac{\partial \zeta_M}{\partial n} = 0 & \text{on} & \Gamma_N \setminus K \\ \dfrac{\partial \zeta_M}{\partial n} + M\zeta_M = 0 & \text{on} & \gamma. \end{array} \tag{27}$$

Setting $\omega_M = \min_{x\in\kappa} \zeta_M(x)$, from equation (26) comes, by using lemma 3, the following estimate:

$$|f_\varphi - f_\psi|_{0,1,K} \geqslant m_\kappa \omega_M |\varphi - \psi|_{0,1,\kappa}$$

which is the estimate (28) with $c' = \frac{1}{m_\kappa \omega_M}$. □

One serious limitation to this result is that the estimate cannot be obtained on the whole γ. However, this restriction may be removed, and a 'fully monotone' stability result obtained provided that some additional requirements on the data and the Robin coefficient itself be fulfilled. This is the subject of the following corollary.

Corollary 1. *Suppose that* $\overline{\Gamma}_D \cap \overline{\gamma} = \emptyset$*, and moreover, that* $\varphi \in \mathcal{C}^0_0(\gamma)$ *and* $\mathrm{supp}(\phi) \subset \Gamma_N$. M *being some positive constant, there then exists some constant* $c = c(\gamma, M) > 0$ *such that given any pair* $(\varphi, \psi) \in (\Phi'_{ad})^2$ *verifying* $\varphi \leqslant \psi \leqslant M$*, we have*

$$|\varphi - \psi|_{0,1,\gamma} \leqslant c(\gamma, M) |f_\varphi - f_\psi|_{0,1,K}. \tag{28}$$

Proof. Since $\overline{\gamma}$ and $\overline{\Gamma}_D$ do not intersect, it turns out that $\overline{\gamma \cup \Gamma_N}$ is a closed connected neighbourhood of $\overline{\gamma}$. Some open subset γ' of $\partial\Omega$, such that $\overline{\gamma} \subset \gamma'$ and $\frac{\partial u}{\partial n} = 0$ on $\gamma' \setminus \overline{\gamma}$, can therefore be found and the Robin boundary condition may be rewritten as follows:

$$\frac{\partial u}{\partial n} + \tilde{\varphi} u = 0 \qquad \text{on} \quad \gamma'$$

where $\tilde{\varphi}$ is the function φ continued by 0 on $\gamma' \setminus \overline{\gamma}$. $\tilde{\varphi}$ is then a continuous function over γ', according to the fact that $\varphi \in \mathcal{C}^0_0(\overline{\gamma})$, and estimate (28) is provided by theorem 3 since $\overline{\gamma}$ is a compact subset of γ'. □

4. Identification

In this section, we are first going to define, following an idea brought by Kohn and Vogelius [18], a cost function as the energy gap between a so-called 'Neumann solution', which is merely the solution of problem (1) with the prescribed flux ϕ as a boundary condition on Γ_N, and a so-called 'Dirichlet' one, which is computed by taking advantage of the measured temperature f. Moreover, let us assume that f has been measured on the whole support of ϕ.

Once again we shall be using the set Φ_{ad} of admissible Robin coefficients as defined in section 2 by (5). Let $\xi \in \Phi_{ad}$, and define $u^0_{N,\xi}$ as the solution of problem (1) with ξ as a Robin coefficient, whereas $u^0_{D,\xi}$ is solution of the following problem:

$$\begin{array}{lll} \Delta u = 0 & \text{in} & \Omega \\ u = 0 & \text{on} & \Gamma_D \\ u = f & \text{on} & \operatorname{supp}(\phi) \\ \dfrac{\partial u}{\partial n} = 0 & \text{on} & \Gamma_N \setminus \operatorname{supp}(\phi) \\ \dfrac{\partial u}{\partial n} + \xi u = 0 & \text{on} & \gamma. \end{array} \tag{29}$$

The cost function $\mathcal{J}$ is therefore defined on Φ_{ad} by

$$\mathcal{J}(\xi) = \int_\Omega |\nabla u^0_{N,\xi} - \nabla u^0_{D,\xi}|^2 + \int_\gamma \xi |u^0_{N,\xi} - u^0_{D,\xi}|^2. \tag{30}$$

Proposition 2. *There exists a unique function $\varphi \in \Phi_{ad}$ such that:*

$$\mathcal{J}(\varphi) \leqslant \mathcal{J}(\psi) \qquad \forall \psi \in \Phi_{ad}.$$

Moreover, φ is the solution of the inverse problem (2).

Proof. Let φ be the solution of our inverse problem (2). Then, $u^0_{N,\varphi}|_K = f$, and thus $u^0_{N,\varphi} = u^0_{D,\varphi}$. φ is therefore a minimum for $\mathcal{J}$ with $\mathcal{J}(\varphi) = 0$.

Now let $\varphi_1 \in \Phi_{ad}$ be another minimum for $\mathcal{J}$. Then $\mathcal{J}(\varphi_1) = 0$, and $u^0_{N,\varphi_1} = u^0_{D,\varphi_1}$. It follows that φ_1 is another solution for the inverse problem (2), which by the identifiability theorem 1 leads to $\varphi_1 = \varphi$. ☐

According to the above proposition, our inverse problem (2) is now turned into the following optimization one:

$$\text{Find } \varphi \in \Phi_{ad} \text{ such that: } \mathcal{J}(\varphi) \leqslant \mathcal{J}(\psi) \qquad \forall \psi \in \Phi_{ad}. \tag{31}$$

To solve such a problem numerically, several algorithms are available. In order to use the gradient method, we need to compute the derivative of the cost function with respect to the unknown, i.e. the Robin coefficient. However, the Gateaux derivative is enough since we need, after problem (1) has been discretized, to compute at each iteration the derivative of the cost function in its gradient direction.

Let $\varphi \in \Phi_{ad}$ and $\psi \in \Phi_{ad}$. Given a small enough real number h, therefore, let u^h_D be the solution of problem (29) with $\varphi^h = \varphi + h\psi$ as a Robin coefficient. By the same techniques used in proposition 1, we get an asymptotic expansion of the solution u^h_D with respect to the parameter h:

$$u^h_D = u^0_D + h u^1_D + h\eta(h) \tag{32}$$

where u^1_D and $\eta(h)$ are elements of $H^1(\Omega)$ such that:

- $\lim_{h\to 0} |\eta(h)|_{1,\Omega} = 0$
- u_D^1 is a solution of the following variational problem:

$$\text{Find } u_D^1 \text{ in } V' \text{ such that} \quad \int_\Omega \langle \nabla u_D^1, \nabla v\rangle + \int_\gamma \varphi u_D^1 v = -\int_\gamma \psi u_D^0 v \qquad \text{for all } v \in V'. \tag{33}$$

- $V' = \{v \in H^1(\Omega) \text{ such that } v|_{\Gamma_D \cup \mathrm{supp}(\phi)} = 0\}$.

The following theorem then holds.

Theorem 4 (derivative of the cost function). *The Gateaux derivative of the cost function at point φ in the ψ-direction, is given by*

$$\mathcal{J}^1(\varphi).\psi := \lim_{h\to 0} \frac{\mathcal{J}(\varphi^h) - \mathcal{J}(\varphi)}{h} - \int_\gamma \psi[(u_D^0)^2 - (u_N^0)^2].$$

Proof. Let $\xi \in \Phi_{ad}$ be any admissible Robin coefficient. Therefore:

$$\mathcal{J}(\xi) = \mathcal{J}_N(\xi) + \mathcal{J}_D(\xi) + \mathcal{J}_{ND}(\xi) \tag{34}$$

where

$$\begin{aligned} \mathcal{J}_N(\xi) &= \int_\Omega |\nabla u^0_{N,\xi}|^2 + \int_\gamma \xi u^0_{N,\xi}{}^2 \\ \mathcal{J}_D(\xi) &= \int_\Omega |\nabla u^0_{D,\xi}|^2 + \int_\gamma \xi u^0_{D,\xi}{}^2 \\ \mathcal{J}_{ND}(\xi) &= -2\left\{ \int_\Omega \langle \nabla u^0_{N,\xi}, \nabla u^0_{D,\xi}\rangle + \int_\gamma \xi u^0_{N,\xi} u^0_{D,\xi} \right\}. \end{aligned} \tag{35}$$

Let us denote by $\mathcal{J}_N^1(\varphi)\cdot\psi$, $\mathcal{J}_D^1(\varphi)\cdot\psi$ and $\mathcal{J}_{ND}^1(\varphi)\cdot\psi$, respectively, the Gateaux derivatives of $\mathcal{J}_N$, $\mathcal{J}_D$ and $\mathcal{J}_{ND}$ at point φ in the ψ-direction.

Integrating by parts, we easily derive that

$$\mathcal{J}_{ND}(\xi) = -2\int_{\Gamma_N} \phi f$$

and therefore that $\mathcal{J}_{ND}^1 = 0$, so that $\mathcal{J}^1(\varphi)\cdot\psi = \mathcal{J}_N^1(\varphi)\cdot\psi + \mathcal{J}_D^1(\varphi)\cdot\psi$.

A straightforward calculation, using the asymptotic expansion of u_N^h, gives

$$\mathcal{J}_N^1(\varphi)\cdot\psi = 2\left\{ \int_\Omega \langle \nabla u_N^1, \nabla u_N^0\rangle + \int_\gamma \varphi u_N^1 u_N^0 \right\} + \int_\gamma \psi (u_N^0)^2$$

and, in the same way

$$\mathcal{J}_D^1(\varphi)\cdot\psi = 2\left\{ \int_\Omega \langle \nabla u_D^1, \nabla u_D^0\rangle + \int_\gamma \varphi u_D^1 u_D^0 \right\} + \int_\gamma \psi (u_D^0)^2.$$

Using equation (12) with u_N^0 as a test function, we get

$$\mathcal{J}_N^1(\varphi)\cdot\psi = -\int_\gamma \psi (u_N^0)^2. \tag{36}$$

On the other hand, integrating by parts, we obtain

$$\int_\Omega \langle \nabla u_D^1, \nabla u_D^0\rangle + \int_\gamma \varphi u_D^1 u_D^0 = \int_{\partial\Omega} \frac{\partial u_D^0}{\partial n} u_D^1 + \int_\gamma \varphi u_D^1 u_D^0$$

which, according to the boundary conditions verified by u_D^0 and u_D^1 gives

$$\int_\Omega \langle \nabla u_D^1, \nabla u_D^0 \rangle + \int_\gamma \varphi u_D^1 u_D^0 = 0$$

Therefore, $\mathcal{J}_D^1(\varphi) \cdot \psi = \int_\gamma \psi (u_D^0)^2$, which together with (36) finally yields

$$\mathcal{J}^1(\varphi) \cdot \psi - \int_\gamma \psi[(u_D^0)^2 = (u_N^0)^2].$$

□

5. Discussion

The Robin direct boundary value problem (1) we have been dealing with in this paper may either arise in thermal imaging, or in corrosion detection by electrostatic measurements; in the first case, the Robin coefficient φ is the heat-exchange coefficient while in the second one, it provides quantitative information on the corrosion occuring on γ. We have proved an identifiability result valid in both 2D and 3D situations. Moreover, this result is almost optimal with respect to the choice of the set of admissible Robin coefficients. Actually, optimality would be achieved by dropping the continuity requirement on φ, since in that case, Φ_{ad} would be the largest set ensuring well posedness for the direct problem. Restricting ourselves to non-negative Robin coefficients, we could prove uniqueness by using the positivity result of lemma 2 instead of the continuity of φ. However, this does not help much, since continuity is still needed for the proof of local Lipschitz stability.

The monotone stability result, even though providing a valuable estimate in somewhat realistic situations, does not contain the local Lipschitz stability one. This fact is helpful, especially when dealing with noisy data, which cannot be expected to fulfil any order relationship with the actual ones. Addressing stability as well as identification issues, the Gateaux differentiation with respect to the coefficient, by providing a variational characterization of the state derivative, once again proves to be a powerful tool.

The monotone result also means that instability can only occur from oscillatory data. A regularizing term in the functional might therefore be necessary to avoid such instabilities in the numerical experiments. However, the cost function used here is an energy least-squares one, based on fields computed from the measured data, and not only on the measured data on the boundary. Therefore, it is expected to be more sensitive to boundary oscillations if any. Previous experiments with similar cost functions have shown efficiency, as well as stability, in several situations, including nonlinear ones ([4, 8, 21], etc).

Theorem 4 opens the road to the implementation of a numerical minimization algorithm using the gradient method, which still needs to be thoroughly tested to provide answers to the questions raised above.

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