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A stable recovery method for the Robin inverse problem

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This paper is dedicated to Rainer Kress for his 60th birthday

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Abstract

We consider the inverse problem of identifying a Robin coefficient by performing measurement on some part of the boundary. After turning the inverse problem to an optimisation one by using a Kohn and Vogelius cost function, we study the stability of this method and present some numericals experiments using synthetic data.

Keywords: Inverse problem; Robin coefficient; Stability; Recovery algorithms

1. Introduction

We consider in this paper the inverse problem of identifying a Robin coefficient φ by impedance tomography using boundary measurements. The experience consists in putting a current flux ϕ on some part Γ_N of the boundary, and thus measuring the electrostatic potential f on some part M of the accessible boundary Γ_N. The inverse problem consists in determining the impedance coefficient φ which characterizes the corrosion level in the metal, from both the knowledge of the prescribed current flux ϕ and the measured potential f.

Several mathematical models regarding the studied phenomenon are met in the literature. The first work goes up to Kaup et al. [11], who modelled the corrosion effects by material losses, resulting from the deterioration of metal, which leads to a modification of the geometry. In further works, Santosa and coworkers [12,13] reduce the knowledge of the damage to that of its effects on the impedance condition—actually a Robin one in the simplest linear case—which holds on the corroded part of the boundary. Determining the Robin coefficient thus would provide with relevant information on the damage.

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This inverse problems has been studied by several authors, who investigated the identifiability and stability issues, and proposed algorithms for the identification. Inglese [8] has proved in a 2D situation that a single measurement on u is enough to determine the Robin coefficient within a class of smooth enough ones (actually C^3 ones), and this class has been later on extended to continuous coefficients in ([4], 1999). Stability in the framework of a thin plate has also been proved in [8,9], whereas a local and directional Lipschitz stability result, as well as a global monotone Lipschitz one, have been obtained in [4]. Isotropic stability has recently been investigated in [1,3,7], who both obtained more or less global stability results of logarithmic type.

As for the identification issue, which is the aim of the present paper, Fasino and Inglese [10] have described and implemented, in the case of a thin plate, a recovery algorithm, based on the asymptotic expansion of the Robin coefficient with respect to the thickness of the plate. Analytic data extension has also been proved in [6] to provide with a robust algorithm for the unknown impedance recovery.

Our concern in the present work is to mathematically and numerically study the features of the algorithm based on the minimization of the so-called Kohn and Vogelius cost function, which has been proposed in [4] for the Robin problem after having been used more than once for various inverse problems. This function is the energetic discrepancy between the solution computed by using the prescribed flux ϕ, and the one computed by taking advantage of the measured potential f. Provided the data are exact (and hence compatible), the minimum of that functional is zero, and its minimum argument is nothing but the inverse problem solution. The situation is somewhat different when measurement errors make the data incompatible. In such a case, the inverse problem has no solution, although the minimization one might. We first prove that the approximate minimization problem has at least one solution. Picking up any of these for a given sequence of data that converge to the actual ones in $H^{1/2}(M)$, we also prove that the so associated sequence of coefficients converge to that of the exact inverse problem in $L^2(\gamma)$. This stability result is—up to our knowledge—original, although robustness has been repeatedly observed in numerical trials. This allows us to claim this algorithm is self stabilizing, and does not therefore need any additional regularization. The robustness issue, meaning the behaviour of the recovered solutions with respect to additional noise that are in $L^2(M)$ rather than in $H^{1/2}(M)$, however still needs to be studied.

Provided the data to recover are smooth, this stabilization feature is usually appreciated. In our case, oscillations of the solution might however be of some interest to locate the corroded parts, and one would not like them erased. Using a gradient algorithms based on the computed expression of the cost function derivative, we observe that its components fastly decrease with respect to the frequencies. There is thus no chance to recover by such an algorithm more than the mean value of the Robin coefficient. This is the reason why we have been preferring a relaxation method on the Fourier basis, in order to recover in turn each of the coefficient components. Still, magnifying the higher frequencies components by using an anti-dumping procedure remains necessary, although the best way to perform it remains an open issue.

The outline of the paper is the following. In Section 2, we recall the problem and the main results regarding identifiability, stability and identification, as established in [4]. Section 3 is devoted to the stability of the minimization algorithm based on the Kohn and Vogelius cost function. Existence of at least one solution to the approximated minimization problem is proved, as well as continuity of the so recovered coefficient with respect to the measured data. Section 4 is finally devoted to the description of the identification process, and to the presentation of the numerical results it provides us with.

2. The Robin inverse problem

Let Ω be a connected bounded domain of $\mathbb{R}^2$. The boundary $\partial\Omega$ is assumed to be a $C^{1,\beta}$ Jordan curve, for some $\beta \in]0, 1$. Moreover, let γ and $\Gamma_{\rm N}$ be two non empty open subsets of $\partial\Omega$ such that:

$$\partial\Omega = \bar{\gamma} \cup \bar{\Gamma}_{\rm N}$$

The inverse problem ($\mathcal{IP}$) we are dealing with is the following:

$$(\mathcal{IP}) \begin{cases} \text{Being given a prescribed flux } \phi \not\equiv 0 \text{ together with measurements } f \text{ on } M\text{, find a function} \\ \quad \varphi \text{ on } \gamma \text{ such that the solution } u \text{ of} \\ \quad (\text{NP}) \begin{cases} \Delta u = 0 \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n} = \phi \text{ on } \Gamma_{\rm N} \\ \dfrac{\partial u}{\partial n} + \varphi u = 0 \text{ on } \gamma \end{cases} \\ \text{also satisfies } u|_M = f \end{cases}$$

2.1. Some previous results regarding the inverse problem

In the sequel, we assume that $\phi \in L^2(\Gamma_{\rm N})$ and φ belongs to a slightly restricted set of admissible parameters $\Phi_{\rm ad}$, defined by:

$$\Phi_{\rm ad} = \{\varphi \in H^1(\gamma), \text{ such that } ||\varphi||_{1,\gamma} \leq c \text{ and } \varphi \geq c'\chi_K\}$$

where c and c' are tow positive constants, and K is a nonempty connected open subset of γ such that $\partial\gamma \cap K = \phi$.

The forward problem (NP) has thus a unique solution in the Hilbert space $H^1(\Omega)$, and referring to [4], the unknown Robin coefficient φ is uniquely determined in $\Phi_{\rm ad}$ from the knowledge of the prescribed current flux ϕ and the measurement data f on M.

Furthermore, a local and directional Lipschitz stability result has been proved in [4], as well as a global monotone one.

- *Local and directional Lipschitz stability*: Let φ and ψ be admissible impedances, $\varphi_h = \varphi + h\psi$ for a small $h > 0$ and u, u_h be the related potentials. Then:

$$\lim_{h\to 0} \frac{|u - u_h|_{0,M}}{h} > 0$$

- *Global monotone stability*: Let $0 < \underline{m} < \bar{m}$. For any subset $\mathcal{K} \subset \gamma$ such that $\mathcal{K} \cap \partial\gamma = \phi$ there exists some positive constant c such that, φ and ψ be a pair of impedances in $\Phi_{\rm ad}$ such that $\underline{m} \leq \varphi \leq \psi \leq \bar{m}$, and u_φ, u_ψ being the related potentials, we have:

$$|\varphi - \psi|_{1,\mathcal{K}} \leq c|u_\varphi - u_\psi|_{1,M}$$

2.2. The Kohn and Vogelius cost function

For sake of simplicity, we shall from now on assume that $M = \Gamma_N$.

For $\varphi \in \Phi_{ad}$, we denote by $u^D(\varphi, f)$ the solution of the following Robin–Dirichlet problem (DP) using the measurements f as a Dirichlet data:

$$\text{(DP)} \begin{cases} \Delta u = 0 \text{ in } \Omega \\ u = f \text{ on } \Gamma_N \\ \dfrac{\partial u}{\partial n} + \varphi u = 0 \text{ on } \gamma \end{cases}$$

We denote also by $u^N(\varphi)$ the solution of the Neumann problem (NP) associated to φ. The solution of the inverse problem $(\mathcal{IP})$ will be denoted by $\bar{\varphi}$.

Let us now define the cost function J on Φ_{ad} by:

$$J(\varphi) = \int_\Omega |\nabla u^N(\varphi) - \nabla u^D(\varphi, f)|^2 + \int_\gamma \varphi |u^N(\varphi) - u^D(\varphi, f)|^2$$

Referring to [4], the function J has a unique minimum which is nothing but the solution $\bar{\varphi}$ of the inverse problem $(\mathcal{IP})$, which is thus turned into the following optimization one:

$$(\mathcal{OP}) \begin{cases} \text{Find } \varphi \in \Phi_{ad} \text{ such that} \\ J(\varphi) \leqslant J(\xi) \quad \forall \xi \in \Phi_{ad}. \end{cases}$$

In order to solve the above problem using a descent method, we need to compute the derivative of the cost function J with respect to the unknown Robin coefficient φ

Proposition 1. [4] Let $\varphi, \psi \in \Phi_{ad}$ *and for a small enough* $h > 0$, *let* $\varphi^h = (\varphi + h\psi)$. *Then we have*:

$$\lim_{h \to 0+} \frac{J(\varphi^h) - J(\varphi)}{h} = \int_\gamma \psi[(u^D(\varphi, f))^2 - (u^N(\varphi))^2]$$

Thanks to this result, we are able to carry out a gradient algorithm in order to solve the optimization problem $(\mathcal{OP})$. At each step of the algorithm, we need to compute the Robin–Dirichlet solution $u^D(\varphi, f)$, and the Robin–Neumann one $u^N(\varphi)$, but no additional adjoint problem is needed in order to compute the gradient of the cost function.

3. Stability of the Kohn and Vogelius method

Although the inverse problem $(\mathcal{IP})$ is not stable, we are going to establish in this section that the optimization one $(\mathcal{OP})$ is. This feature is due to the fact that the Kohn and Vogelius cost function involves the solutions computed from both the prescribed and measured data through their values inside the domain, and not only on the boundary. Therefore, unstable behaviours away from the prescription part of the boundary are prohibited since they would dramatically impact the cost function.

Let us denote by f_n a sequence of “measurements” in $H^{1/2}(M)$, such that

$$\lim_{n\to\infty} ||f_n - f||_{1/2,M} = 0$$

Actually, measurements would rather belong to $L^2(M)$ than to $H^{1/2}(M)$, and f_n may thus be seen as smooth perturbations of the actual data *f*. Two issues here need to be addressed:

- The first one is to figure out whether the minimisation problem $(\mathcal{OP})$ still has a solution φ_n when the Dirichlet data *f* is replaced by f_n on $\Gamma_{\rm N}$. The point is that the pair (ϕ, f_n) is no longer *compatible*, meaning that the inverse problem related to that pair has thus no solution. The optimization problem $(\mathcal{OP}_n)$ related to the same pair is therefore no longer equivalent to the inverse problem, but what about its solutions? Provided it has any, the minimum of the cost function *J* would of course no longer be zero. And it also might be attained by more than a single argument.
- Provided the perturbated optimization problem has indeed solutions, can we prove a convergence result such as $\lim_{n\to\infty} \varphi_n = \bar{\varphi}$ in some appropriated space?

To check these issues, let us define the following function:

$$\begin{aligned}&\zeta : \Phi_{\rm ad} \times H^{1/2}(\Gamma_{\rm N}) \to \mathbb{R}\\ &(\varphi, g) \mapsto \int_\Omega |\nabla u^{\rm D}(\varphi, g) - \nabla u^{\rm N}(\varphi)|^2 + \int_\gamma \varphi |u^{\rm D}(\varphi, g) - u^{\rm N}(\varphi)|^2\end{aligned}$$

where $u^{\rm D}(\varphi, g)$ denotes the solution of the (DP) problem with φ as a Robin coefficient, and *g* as Dirichlet boundary data on $\Gamma_{\rm N}$ and $u^{\rm N}(\varphi)$ is as usual solution of the (NP) problem with φ as a Robin coefficient.

Lemma 2. *Let* $g \in H^{1/2}(\Gamma_{\rm N})$. *Then, there exists* $\varphi^g \in \Phi_{\rm ad}$ *such that*:

$$\inf_{\varphi\in\Phi_{\rm ad}} \zeta(\varphi, g) = \zeta(\varphi^g, g).$$

Proof. Since $\zeta(\varphi, g) \geq 0\, \forall g \in H^{1/2}(\Gamma_{\rm N})$ let $\lambda^g \in \mathbb{R}^+$ be the infimum of that function with respect to φ:

$$\lambda^g = \inf_{\varphi\in\Phi_{\rm ad}} \zeta(\varphi, g).$$

and let φ_n^g a minimizing sequence in $\Phi_{\rm ad}$, thus verifying:

$$\lambda^g = \lim_{n\to+\infty} \zeta(\varphi_n^g, g)$$

The sequence $(\varphi_n^g)_n$ is therefore bounded in $H^1(\gamma)$, and there exists a subsequence of it, still denoted (φ_n^g), such that:

$$\begin{cases}\varphi_n^g \to \varphi^g \text{ weakly in } H^1(\gamma)\\ \varphi_n^g \to \varphi^g \text{ strongly in } L^2(\gamma)\end{cases} \tag{1}$$

Being a proper closed convex subset of $H^1(\gamma)$, $\Phi_{\rm ad}$ is also a closed subset of $H^1(\gamma)$ for the weak topology [2], from which we derive that φ^g belongs to $\Phi_{\rm ad}$. □

According to [4], the mapping:

$$\begin{aligned}&L^2(\gamma) \mapsto H^1(\Omega)\\ &\varphi \mapsto u^N(\varphi)\end{aligned} \tag{2}$$

is continuous with respect to the strong topology, and hence:

$$\lim_{n\to\infty} u^{\mathrm{N}}(\varphi_n^g) = u^{\mathrm{N}}(\varphi^g) \text{ in } H^1(\Omega) \tag{3}$$

Let us now prove that $\lim_{n\to\infty} u^{\mathrm{D}}(\varphi_n^g, g) = u^{\mathrm{D}}(\varphi^g, g)$ in $H^1(\Omega)$.The function $w_n = u^{\mathrm{D}}(\varphi_n^g, g) - u^{\mathrm{D}}(\varphi^g, g)$ solves the following boundary problem:

$$(\mathrm{P})\begin{cases} \Delta w_n = 0 \text{ in } \Omega \\ w_n = 0 \text{ on } \Gamma_N \\ \dfrac{w_n}{\partial n} + \varphi_n^g w_n = -(\varphi_n^g - \varphi^g)u^{\mathrm{D}}(\varphi^g, g) \text{ on } \gamma \end{cases}$$

Problem (P) has a unique solution in the set:

$$V = \{v \in H^1(\Omega); \quad v = 0 \text{ on } \Gamma_{\mathrm{N}}\},$$

and the variational formulation of its is:

$$\begin{cases} \text{Find } w_n \in V \text{ such that} \\ \int_\Omega \nabla w_n \nabla v + \int_\gamma \varphi_n^g w_n v = -\int_\gamma (\varphi_n^g - \varphi^g)u^{\mathrm{D}}(\varphi^g, g)v \qquad \forall v \in V. \end{cases}$$

Setting $v = w_n$, we obtain:

$$\int_\Omega |\nabla w_n|^2 + \int_\gamma \varphi_n^g (w_n)^2 = -\int_\gamma (\varphi_n^g - \varphi) w_n u^{\mathrm{D}}(\varphi^g, g) \tag{4}$$

The sequence φ_n^g belongs to Φ_{ad}, then

$$\int_\Omega |\nabla w_n|^2 + \int_\gamma \varphi_n^g (w_n)^2 \geq \int_\Omega |\nabla w_n|^2 + c' \int_K (w_n)^2. \tag{5}$$

On the other hand, the map $v \mapsto (\int_\Omega |\nabla v|^2 + c' \int_K v^2)^{1/2}$ defines a norm on H^1, equivalent to the classical H^1 norm, then there exists some constant $\beta > 0$ such that:

$$||v||_{1,\Omega}^2 \leq \beta \left[\int_\Omega |\nabla v|^2 + c' \int_K v^2 \right] \qquad \forall v \in H^1(\Omega) \tag{6}$$

By using Eqs. (4)–(6), we get:

$$||w_n||_{1,\Omega}^2 \leq \beta ||\varphi_n^g - \varphi^g||_{\infty,\gamma} ||w_n||_{0,\gamma} ||u^{\mathrm{D}}(\varphi^g, g)||_{0,\gamma}.$$

and then:

$$||w_n||_{1,\Omega} \leq \beta\alpha ||u^{\mathrm{D}}(\varphi^g, g)||_{0,\gamma} ||\varphi_n^g - \varphi^g||_{\infty,\gamma}.$$

where α is the norm of the trace mapping.

According to Eq. (1), and to the compact imbedding from $H^1(\gamma)$ into $C^0(\bar{\gamma})$, there exists a subsequence of $(\varphi_n^g)_n$, still denoted by $(\varphi_n^g)_n$, such that

$$\lim_{n\to+\infty} ||\varphi_n^g - \varphi^g||_{\infty,\gamma} = 0$$

and then:

$$u^{\mathrm{D}}(\varphi_n^g, g) \to u^{\mathrm{D}}(\varphi^g, g) \text{ strongly in } H^1(\Omega) \tag{7}$$

By using Eqs. (3) and (7), we obtain:

$$\lim_{n\to+\infty} \int_\Omega |\nabla u^{\mathrm{D}}(\varphi_n^g, g) - \nabla u^{\mathrm{N}}(\varphi_n^g)|^2 = \int_\Omega |\nabla u^{\mathrm{D}}(\varphi^g, g) - \nabla u^{\mathrm{N}}(\varphi^g)|^2. \tag{8}$$

Moreover, we have:

$$\begin{cases} \varphi_n^g \to \varphi^g \text{ in } L^\infty(\gamma) \\ [u^{\mathrm{D}}(\varphi_n^g, g) - u^{\mathrm{N}}(\varphi_n^g)] \to [u^{\mathrm{D}}(\varphi^g, g) - u^{\mathrm{N}}(\varphi^g)] \quad \text{strongly in } L^2(\gamma) \end{cases}$$

then:

$$\lim_{n\to\infty} \int_\gamma \varphi_n^g |u^{\mathrm{D}}(\varphi_n^g, g) - u^{\mathrm{N}}(\varphi_n^g)|^2 = \int_\gamma \varphi^g |u^{\mathrm{D}}(\varphi^g, g) - u^{\mathrm{N}}(\varphi^g)|^2 \tag{9}$$

Eqs. (8) and (9) then give:

$$\lambda^g = \lim_{n\to+\infty} \zeta(\varphi_n^g, g) = \zeta(\varphi^g, g),$$

and finally:

$$\inf_{\varphi\in\Phi_{\mathrm{ad}}} \zeta(\varphi, g) = \zeta(\varphi^g, g)$$

Lemma 3. *Let* $f_n \in H^{1/2}(\Gamma_{\mathrm{N}})$ *a sequence of data such that* $\lim_{n\to+\infty} ||f_n - f||_{1/2,\Gamma_{\mathrm{N}}} = 0$, *and let* φ_n *be any function of* Φ_{ad} *such that such that* $\inf_{\varphi\in\Phi_{\mathrm{ad}}} \zeta(\varphi, f_n) = \zeta(\varphi_n, f_n)$. *Then we have*:

$$\lim_{n\to+\infty} \zeta(\varphi_n, f_n) = 0$$

Proof. First we have:

$$0 \le \zeta(\varphi_n, f_n) \le \zeta(\bar\varphi, f_n), \tag{10}$$

Let us prove now that $\lim_{n\to\infty} \zeta(\bar\varphi, f_n) = 0$
Let w_n be defined by

$$w_n := u^{\mathrm{D}}(\bar\varphi, f_n) - u^{\mathrm{N}}(\bar\varphi) = u^{\mathrm{D}}(\bar\varphi, f_n) - u^{\mathrm{D}}(\bar\varphi, f)$$

w_n, therefore, solves the boundary value problem:

$$\begin{cases} \Delta w_n = 0 \text{ in } \Omega \\ w_n = f_n - f \text{ on } \Gamma_{\mathrm{N}} \\ \dfrac{\partial w_n}{\partial n} + \bar\varphi w_n = 0 \text{ on } \gamma \end{cases}$$

Since $\lim_{n\to\infty} ||f_n - f||_{1/2,\Gamma_{\mathrm{N}}} = 0$, we get:

$$\lim_{n\to+\infty} ||w_n||_{1,\Omega} = 0 \tag{11}$$

But $\zeta(\bar{\varphi}, f_n) = \int_\Omega |\nabla w_n|^2 + \int_\gamma \bar{\varphi}(w_n)^2$ and hence:

$$\lim_{n\to\infty} \zeta(\bar{\varphi}, f_n) = 0 \tag{12}$$

which, by Eq. (10), gives:

$$\lim_{n\to\infty} \zeta(\varphi_n, f_n) = 0.$$

We are now able to state the stability result. □

Theorem 4 (Stability of the optimization problem). *The sequence φ_n converge to $\bar{\varphi}$ strongly in $L^2(\gamma)$.*

Proof. Let μ be any accumulation point of the real bounded sequence $||\varphi_n - \bar{\varphi}||_{0,\gamma}$. There exists some subsequence of φ_n, still denoted by φ_n such that:

$$\lim_{n\to\infty} ||\varphi_n - \bar{\varphi}||_{0,\gamma} = \mu$$

We also have $\inf_{\varphi\in\Phi_{\text{ad}}} \zeta(\varphi, f_n) = \zeta(\varphi_n, f_n)$. The sequence $(\varphi_n)_n$ is bounded in $H^1(\gamma)$, then, there exists $\underline{\varphi} \in \Phi_{\text{ad}}$ and a subsequence of φ_n still denoted φ_n such that:

$$\varphi_n \rightharpoonup \underline{\varphi} \text{ weakly in } H^1(\gamma)$$
$$\varphi_n \to \underline{\varphi} \text{ strongly in } L^2(\gamma)$$

Let us now prove that $\lim_{n\to\infty} \zeta(\varphi_n, f_n) = \zeta(\underline{\varphi})$. There exists some $\lambda > 0$ such that:

$$\begin{aligned} ||u^{\text{D}}(\varphi_n, f_n) - u^{\text{N}}(\varphi_n)||^2_{1,\Omega} &\le \lambda \left[\int_\Omega |\nabla(u^{\text{D}}(\varphi_n, f_n) - u^{\text{N}}(\varphi_n))|^2 + c' \int_K (u^{\text{D}}(\varphi_n, f_n) - u^{\text{N}}(\varphi_n))^2\right] \\ &\le \lambda\zeta(\varphi_n, f_n) \end{aligned}$$

According to Lemma 3, we have $\lim_{n\to\infty} \zeta(\varphi_n, f_n) = 0$, and then:

$$\lim_{n\to\infty} ||u^{\text{D}}(\varphi_n, f_n) - u^{\text{N}}(\varphi_n)||^2_{1,\Omega} = 0$$

Since $\varphi_n \to \underline{\varphi}$ strongly in $L^2(\gamma)$, we obtain by using Eq. (2):

$$u^{\text{N}}(\varphi_n) \to u^{\text{N}}(\underline{\varphi}) \text{ strongly in } H^1(\Omega). \tag{13}$$

Then,

$$u^{\text{D}}(\varphi_n, f_n) \to u^{\text{N}}(\underline{\varphi}) \text{ strongly in } H^1(\Omega),$$

and therefore:

$$u^{\text{D}}(\varphi_n, f_n)|_{\Gamma_{\text{N}}} \to u^{\text{N}}(\underline{\varphi})|_{\Gamma_{\text{N}}} \text{ in } H^{1/2}(\Gamma_N)$$

Moreover, $u^{\text{D}}(\varphi_n, f_n)|_{\Gamma_N} = f_n$ and $f_n \to f$ in $H^{1/2}(\Gamma)_{\text{N}}$, then :

$$u^{\text{N}}(\underline{\varphi})|_{\Gamma_{\text{N}}} = f.$$

$\underline{\varphi}$ then solves problem $(\mathcal{IP})$, which by uniqueness yields $\underline{\varphi} = \bar{\varphi}$ and then $\mu = 0$.

Thus:

$$\varphi_n \to \bar{\varphi} \text{ strongly in } L^2(\gamma). \qquad \square$$

4. Numerical results

Experimental measurements are simulated by synthetic data obtained by means of numerical computations, solving problem (N_P). Thanks to [4], we are able to compute the gradient of the cost function without need of an adjoint problem solution. A gradient method seems thus the most appropriate to solve the minimization problem ($\mathcal{OP}$). Having chosen an approximation space for the Robin coefficient, the gradient algorithm is the following:

The gradient algorithm applied to the KV cost function:

(1) *Initialisation*: Choose some initial guess φ_0 in the approximation space, and a step size $\rho > 0$;
(2) *Iteration*: φ_k being computed,
   (a) Compute the solutions of u_k^N and u_k^D of the Neumann and Dirichlet problems (P_k^N) and (P_k^D) related to φ_k
   (b) Compute the gradient of J at φ_k, using the following formula giving the derivative of J in any direction ψ:

$$\nabla J(\varphi_k)\psi = \int_\gamma \psi[(u_k^\mathrm{D})^2 - (u_k^\mathrm{N})^2)]$$

   (c) Update the Robin coefficient by:

$$\varphi_{k+1} = \varphi_k - \rho \nabla J(\varphi_k)$$

(3) *Stop test*: If $(|\varphi_{k+1} - \varphi_k|)/|\varphi_k|$ is small enough, then stop, else return to step (2) with $k = k + 1$

Problems (P_k^N) and (P_k^D) have been solved using piecewise quadratic finite elements. At each step, the rigidity matrices need updating, since they expand as follows:

$$A(\varphi_k) = A + a(\varphi_k)$$

where A stands for the invariant parts of the matrices, related to the degrees of freedom located out of γ, whereas $a(\varphi_k)$ is the variable part of them—which is the same one for both the Dirichlet and Neumann matrices—depending on φ_k, and related to the degrees of freedom located on γ.

Solutions of the linear systems related to problems (P_k^N) and (P_k^D) have been obtained using the conjugate gradient algorithm, preconditioned by an incomplete Cholesky factorization of the invariant parts of the matrices, thus preserving their profiles. All the numerical experiments have been carried out on the unit disc, with $\gamma = \{\mathrm{e}^{i\theta}; \theta \in]0, \frac{\pi}{2}[\}$.

4.1. Representation using finite element shape functions

Several options are available to represent the approximated impedance (Robin coefficient). One of them is to use finite element shape functions, not necessarily those used to run the finite element computations of the states u^D and u^N. It is well known that, aiming to solve inverse problems, the unknowns to recover should preferably not be too numerous. A hierarchic approach, gradually enriching the representation, has been to that end successfully experienced in [5] for the recovery of boundaries using the KV cost function. In the present work, a piecewise linear representation of the impedance φ has been preferred to the piecewise quadratic one, although these elements are those used in the finite element computations.

The impedance φ is thus parameterized by a number N_p of unknown $(\varphi_1 \ldots, \varphi_{N_p})$ with

$$\varphi_i = \varphi(\theta_i) \text{ where } \theta_i = \frac{i\pi}{2(N_p+1)}, \quad \text{for } i = 1, 2 \ldots N_p$$

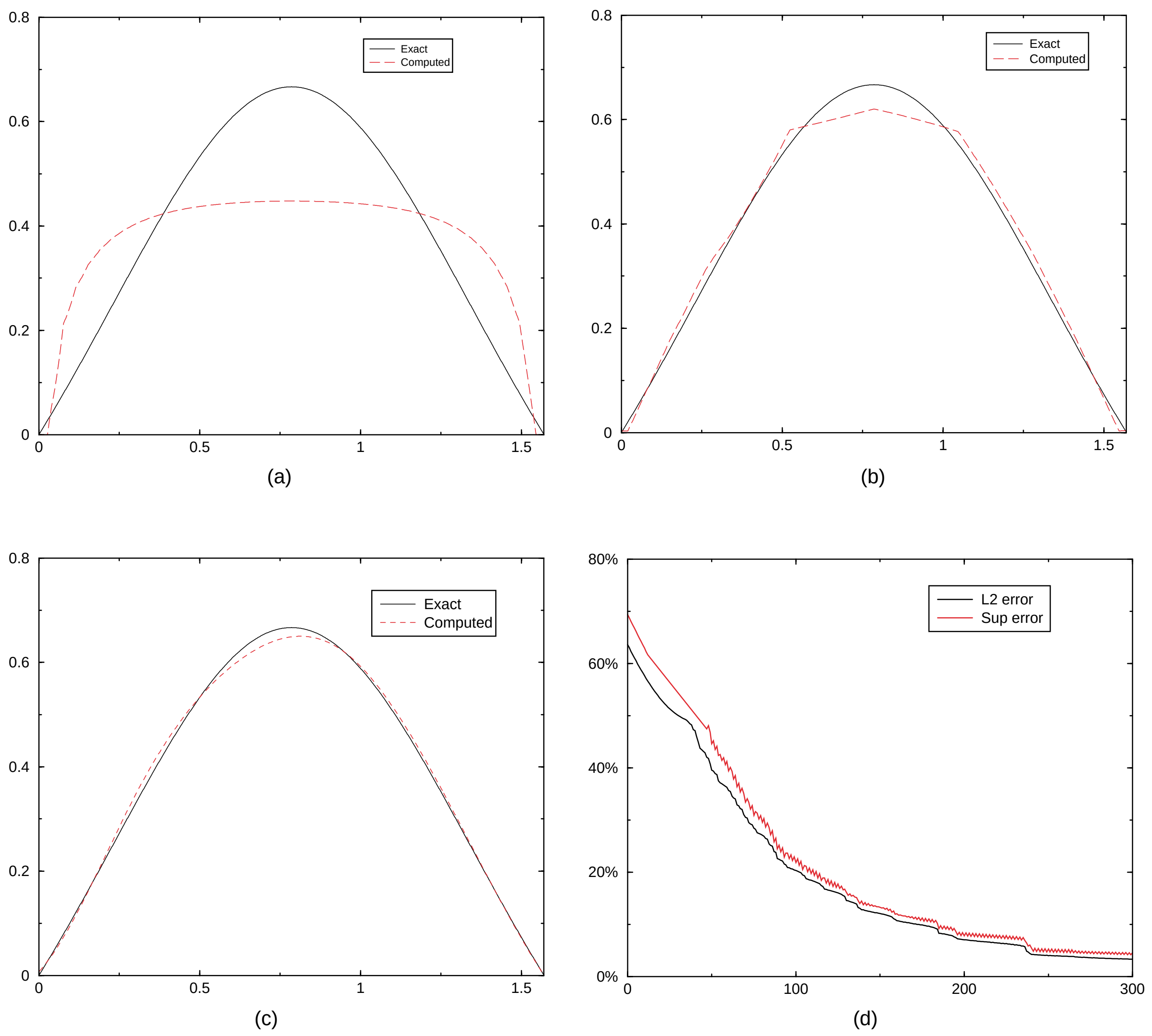

Fig. 1. Reconstruction by piecewise linear representation: (a) gradient (error = 29%); (b) hierarchic gradient (error =5.5%); (c) hierarchic + smoothing (error = 2%); (d) error wrt number of iterations.

and

$$\varphi = \sum_{i=1}^{N_p} \varphi_i \chi_i$$

χ_i being the piecewise linear 'hat' function defined by $\chi_i(\theta_j) = \delta_{ij}; i, j = 1, \dots, N_p$.

Fig. 1 show, as has already been noticed in [5], that a best recovery of smooth impedances is obtained by using a hierarchic approach, together with a smoothing. However, the method fails in fitting non smooth impedances, and above all oscillating ones, as shown in Fig. 2.

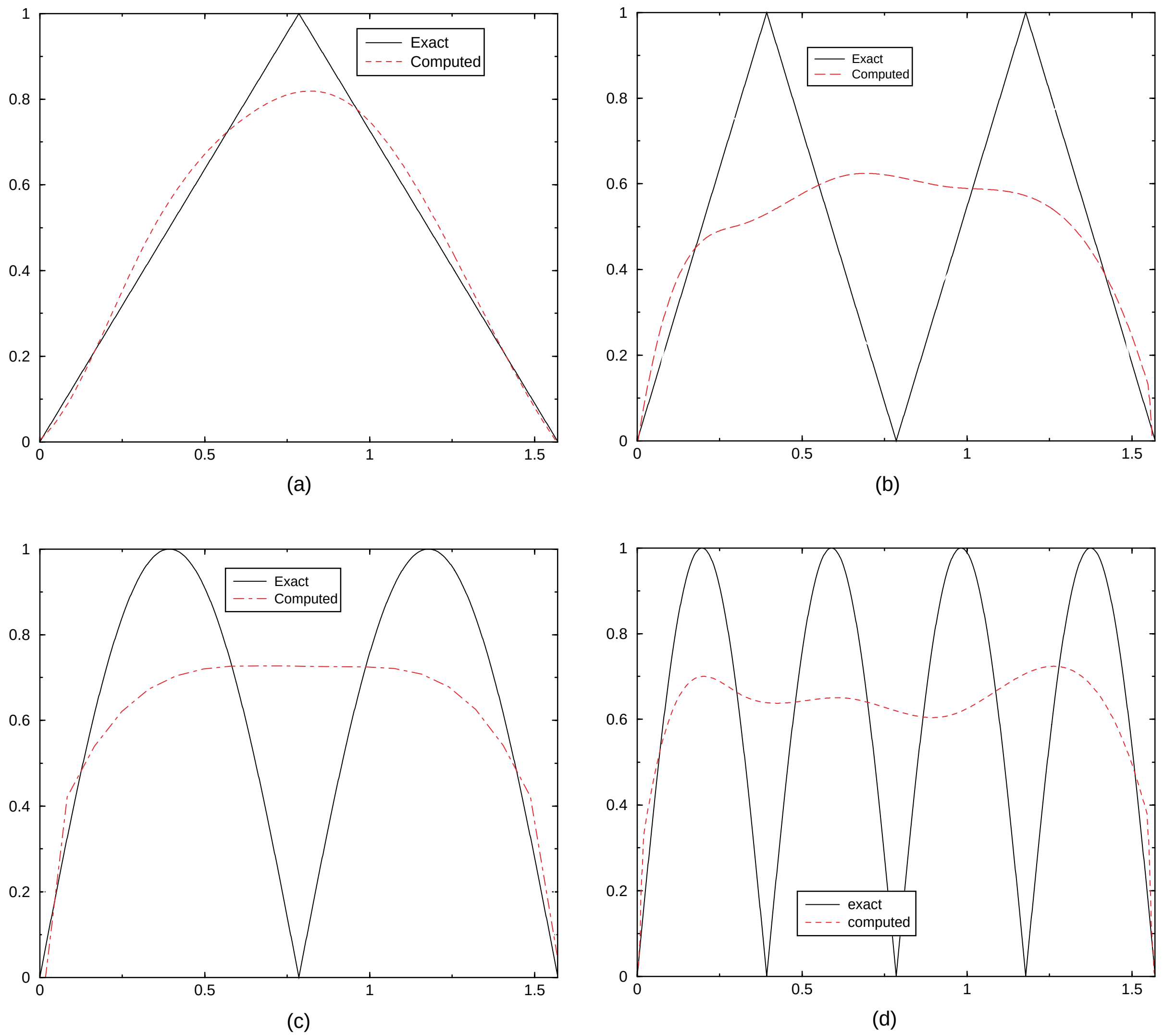

Fig. 2. Reconstruction by piecewise linear representation of non smooth and oscillating impedances. Hierarchic gradient, 5–80 DOF.

4.2. Representation using a Fourier basis

Using a Fourier basis in order to capture the oscillations may thus be viewed as a reliable alternative. To this end, let us expand the impedance φ as follows:

$$\varphi(\theta) = \sum_{j=0}^{\infty}[a_j \cos(2j\theta) + b_j \sin(2j\theta)]$$

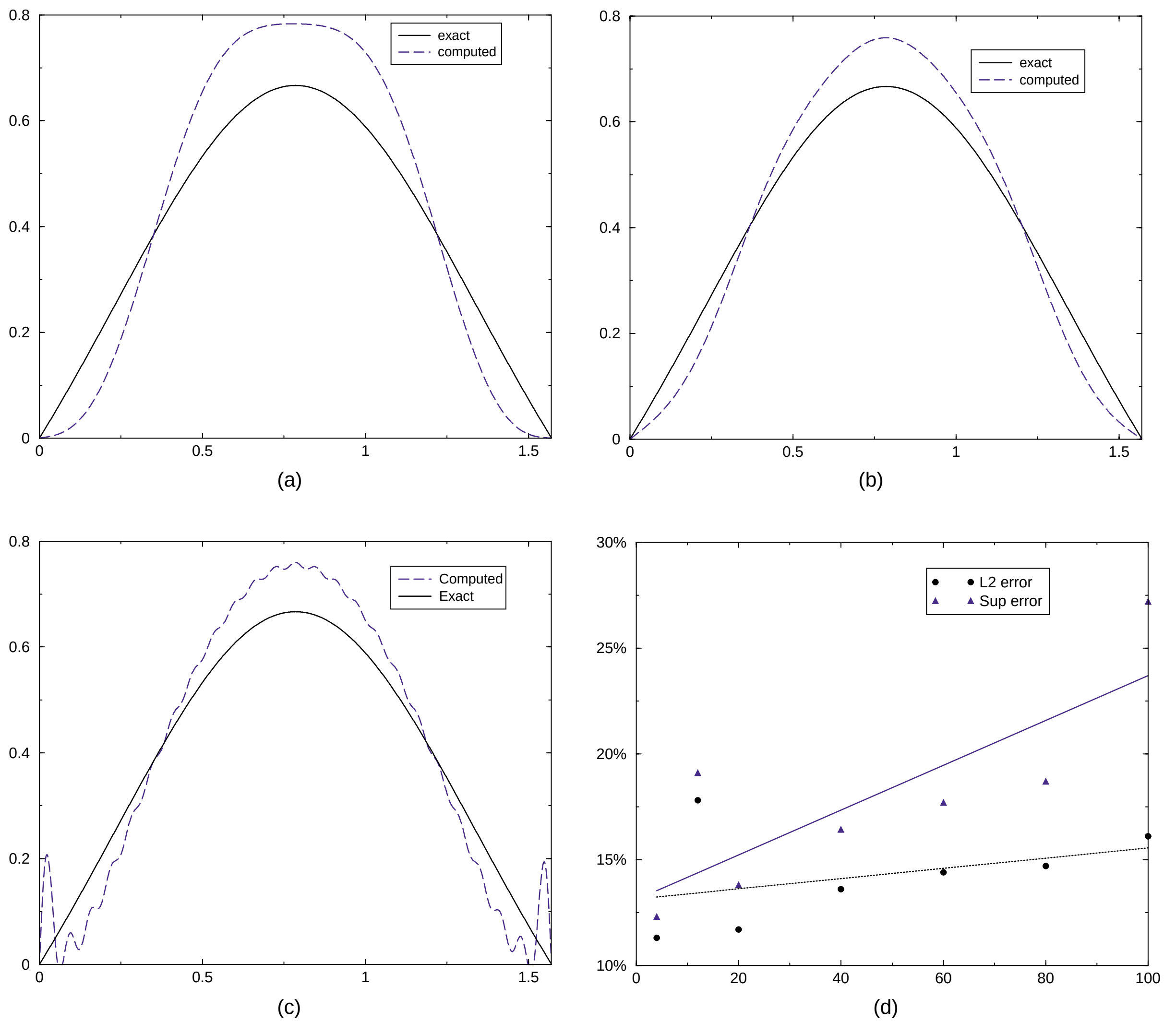

Fig. 3. Reconstruction on a Fourier basis by a gradient method: (a) 12 Fourier coefficients; (b) 20 Fourier coefficients; (c) 100 Fourier coefficients; (d) error wrt number of modes.

and the trucated series is given by

$$\varphi^N(\theta) = \sum_{j=0}^{N}[a_j \cos(2j\theta) + b_j \sin(2j\theta)]$$

Fig. 3 clearly shows that the reconstructed solutions using the gradient algorithm as described above, with the Fourier basis, are polluted by Gibbs effects, and do not provide with satisfactory results even for smooth impedances. The reason why it is so is that the gradient components of the cost function with respect to the higher frequencies basis functions are (for instance for cosines):

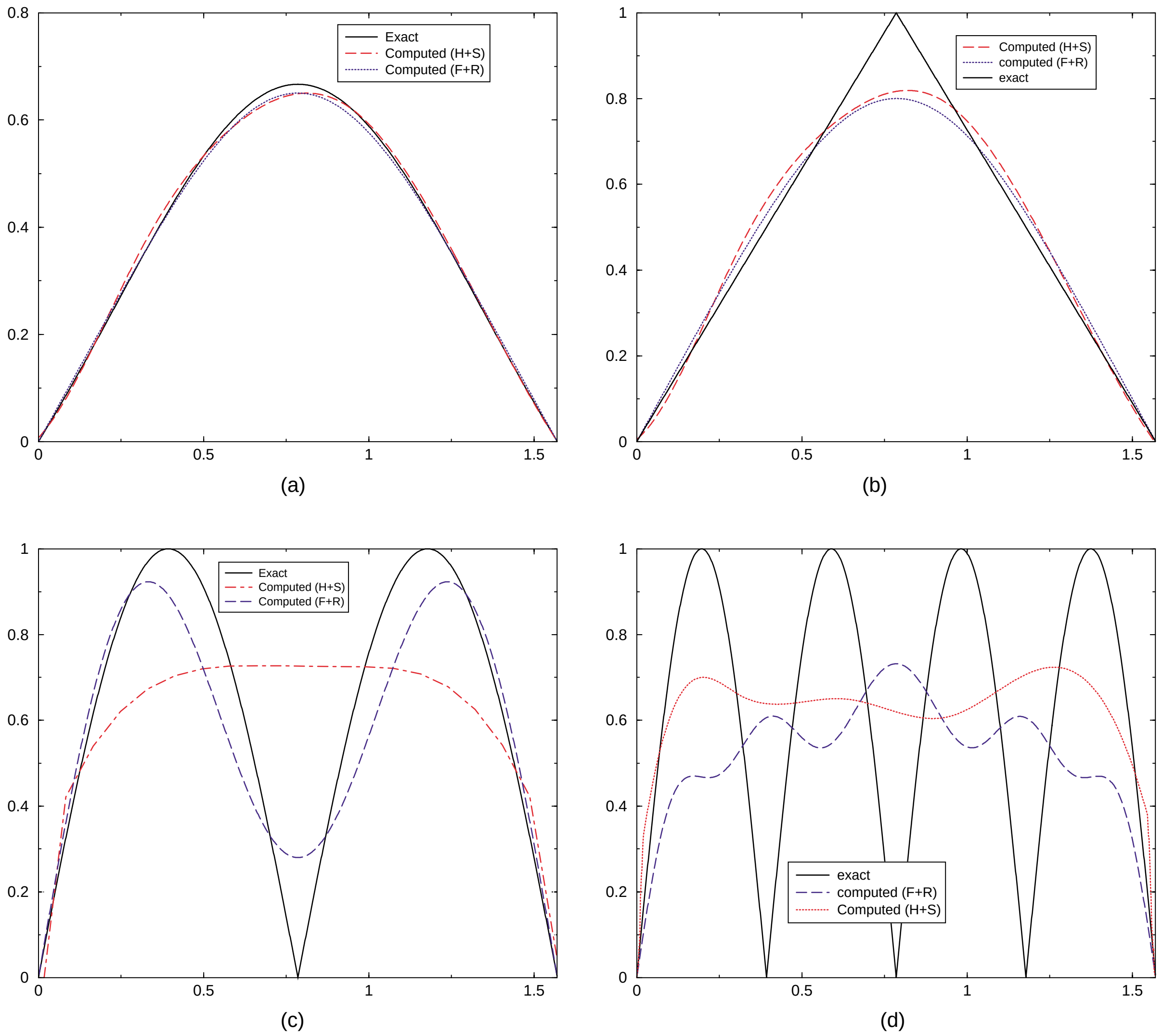

Fig. 4. Reconstruction on a Fourier basis (relaxation method).

$$\frac{\partial J}{\partial \varphi}(\cos(2j\theta)) = \nabla J(\varphi)\cos(2j\theta) = \int_0^{\pi/2} \cos(2j\theta)[(u^{\mathrm{D}})^2 - (u^{\mathrm{N}})^2]\,\mathrm{d}\theta$$

This is nothing but the jth Fourier coefficient of $[(u_{\mathrm{D}})^2-(u_{\mathrm{N}})^2]$, which is fastly decreasing to zero with respect to j if this function is smooth enough. The gradient algorithms thus seeks descent only along the lower frequencies, crushing the components in the higher ones. In order to take care of each of the components, a relaxation method, minimizing successively along all of the frequencies, has thus been preferred to the gradient one. Let N be the number of modes used for the representation:

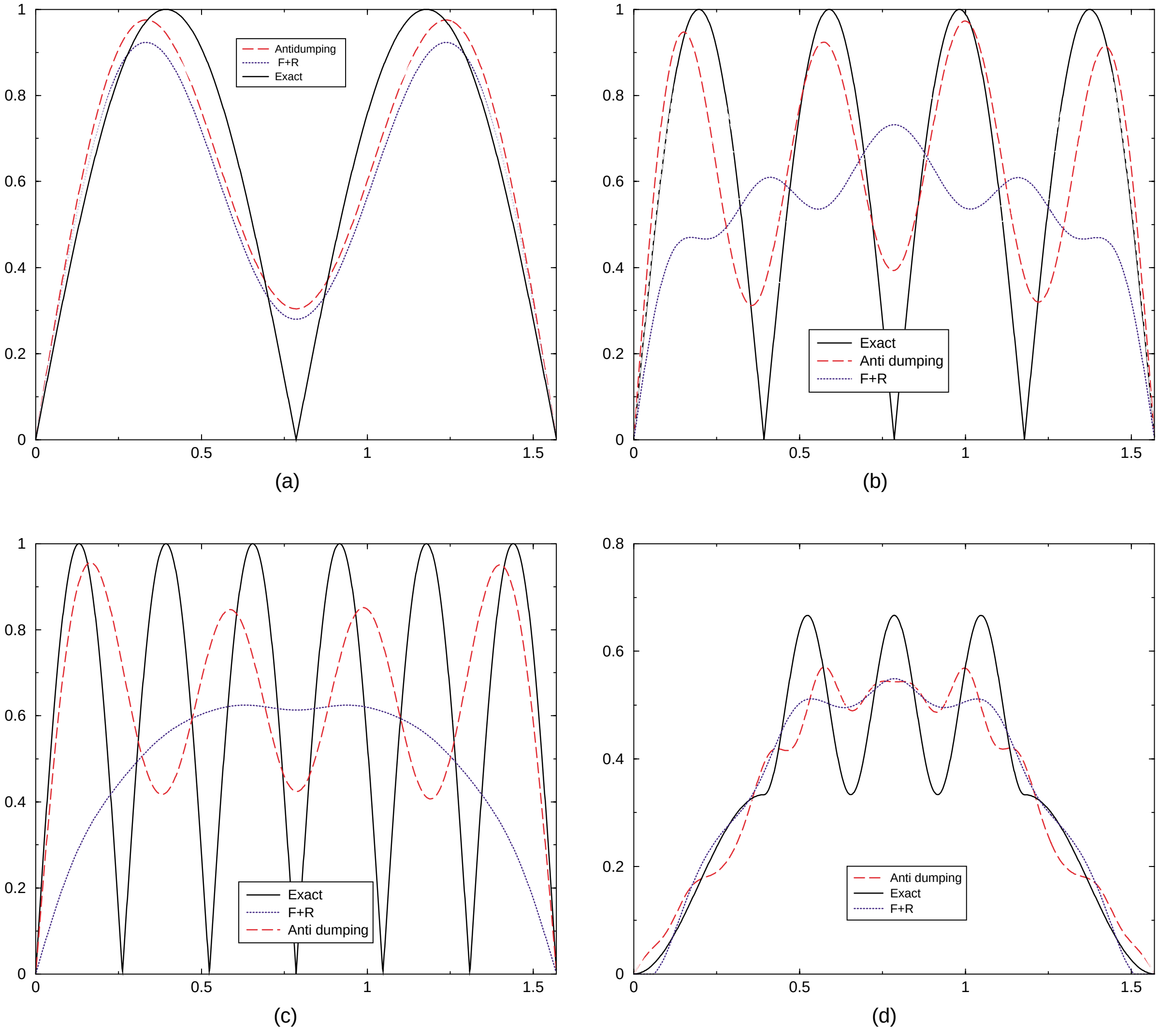

Fig. 5. Capturing oscillations by anti-dumping (Fourier + relaxation).

The relaxation algorithm applied to the KV cost function:

(1) *Initialisation*: Choose some initial guess φ_0 in the approximation space:

$$\varphi_k^0 = \sum_{j=0}^{N} [a_j^0 \cos(2j\theta) + b_j^0 \sin(2j\theta)]$$

(2) *Iteration*: $k+1$: φ_k having been computed:

$$\varphi_k^0 = \sum_{j=0}^{N} [a_j^k \cos(2j\theta) + b_j^k \sin(2j\theta)]$$

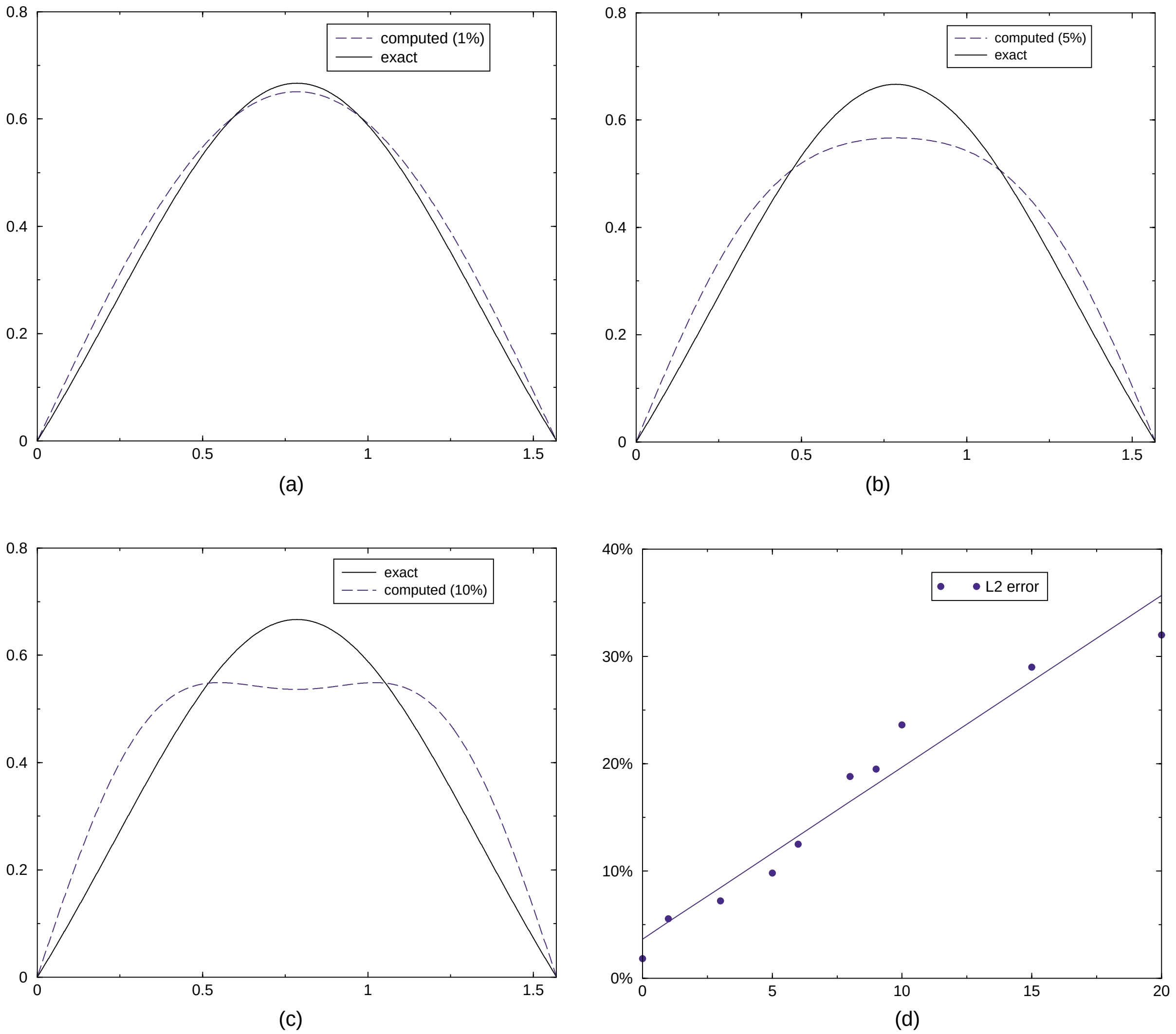

Fig. 6. Noisy data (Fourier + relaxation): (a) 1%; (b) 5%; (c) 10%; (d) error wrt noise level.

Then, from $l = 1, \ldots, N$, $\varphi_{k+(l-1)/N}$ being known, compute $\varphi_{k+l/N}$ by minimizing J along the line $\varphi_{k+(l-1)/N} + \{\cos(2l\theta), \sin(2l\theta)\}$, using the gradient method;

(3) *Stop test*: If $|\varphi_{k+1} - \varphi_k|/|\varphi_k|$ small enough, then stop, else go to step 2 with $k = k + 1$.

This algorithm has been carried out using a hierarchic approach, thus progressively increasing the number of modes used for the representation of the impedances, in order not to handle a too large amount of unknowns.

The method turns out to better capture oscillations, as shown in Fig. 4, but still fails in accurately fitting highly oscillating impedances. A heuristic anti-dumping procedure, consisting to magnify the jth frequency components by a factor $\sqrt{j}$, happens to provide better results, as Fig. 5 shows. However, although interesting, the procedure still needs a thorough study in order to determine the appropriate magnification factor, and overall to prevent the process from generating undesired oscillations.

Fig. 6 finally shows the method to resist quite high levels of noise.

5. Conclusions

We have presented in this paper a stable numerical method to recover the impedance in a Robin problem. Minimizing the Kohn and Vogelius cost function, which is the energetic discrepancy between the Neumann solution computed using the prescribed current flux, and the Dirichlet one obtained by from the measured potential, is proved to be a stable numerical method with respect to the measured data. Its stability might even be excessive, since oscillations of the recovered impedances are dumped whatever relevant they might be with respect to the corrosion phenomenon description. An anti-dumping procedure permits to partly restore these oscillations, but it still needs to be thoroughly studied, and carefully handled in order not to generate artificial oscillations.

Although the robustness of the method has been observed through numerical experiments, the theoretical issue still needs to be investigated. Robustness means that non smooth perturbations of the data, provided they are small, would produce small perturbations in the recovered impedances. Eventually, although the whole work has been performed in a 2D framework, most its theoretical results can be transferred to 3D, where the numerics would however be somewhat more complicated to run.

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