Basic Notions on Graphs

Eulerian and Hamiltonian Graphs; Edge Colouring

Presented by
Joe Ryan
School of Electrical Engineering and Computer Science
University of Newcastle, Australia
Exploring and travelling

**Explorer’s Problem.** An explorer wishes to find a tour that traverses each *road* exactly once and returns to the starting point.

**Traveller’s Problem.** A traveller wishes to find a tour that visits each *city* exactly once and returns to the starting point.
Exploring and travelling

*In graph-theoretical terms:*

**Explorer’s Problem.** Find a *closed trail* that includes every edge of the graph.

**Traveller’s Problem.** Find a *cycle* that includes every vertex of the graph.
Eulerian and Hamiltonian graphs

A connected graph is **Eulerian** if it contains a closed trail that includes every edge; such a trail is an **Eulerian trail**.

A connected graph is **Hamiltonian** if it contains a cycle that includes every vertex; such a cycle is a **Hamiltonian cycle**.

- Eulerian: (a) and (b); Hamiltonian: (a) and (c)
Eulerian and Hamiltonian graphs

**Problem** Decide which of the following graphs are Eulerian and/or Hamiltonian, and write down an Eulerian trail or Hamiltonian cycle, where possible.
Eulerian graphs

**Theorem** Let $G$ be a graph in which each vertex has even degree. Then $G$ can be split into cycles, no two of which have an edge in common.

**Problem** Show how the following graph can be split into cycles, no two of which have an edge in common. How can these cycles be combined to form an Eulerian trail?
Eulerian graphs

**Theorem** A connected graph is Eulerian if and only if each vertex has even degree.

**Problem** Use the Theorem to determine which of the following graphs are Eulerian.

(a) The complete graph $K_8$;
(b) The complete bipartite graph $K_{8,8}$;
(c) The cycle graph $C_8$;
(d) The dodecahedron graph;
(e) The cube graph $Q_8$.

*and so also*

**Theorem** An Eulerian graph can be split into cycles, no two of which have an edge in common.
Dodecahedron and Cubes

The graph $Q_k$ has $2^k$ vertices, and is regular of degree $k$. It follows from Theorem 2.2 that $Q_k$ has $k \times 2^{k-1}$ edges.
Semi-Eulerian graphs

Consider modifications to the idea of ‘being Eulerian’.

A connected graph is semi-Eulerian if there is an open trail that includes every edge; such a trail is a semi-Eulerian trail.

The graph below is not semi-Eulerian.
Semi-Eulerian graphs

**Theorem** A connected graph is semi-Eulerian if and only if it has exactly two vertices of odd degree.

**Proof.** (a) *If G is semi-Eulerian then G has exactly two vertices of odd degree:*

(b) *If a connected graph G has exactly two vertices of odd degree then G is semi-Eulerian:*

![Diagram of a semi-Eulerian graph](image_url)
Problem: Use the Theorem to determine which of the following graphs are semi-Eulerian, and write down a corresponding open trail, where possible.
Dominoes

The game of dominoes can be considered as an application of Eulerian graphs. We use $K_7$ which is Eulerian since each vertex degree is even.

An Eulerian trail is for example

01,12,23,34,45,56,60,02,24,46,61,13,35,50,03,36,62,25,51,14,40

Each edge represents a domino.
Dominoes

The Eulerian trail
01,12,23,34,45,56,60,02,24,46,61,13,35,50,03,36,62,25,51,14,40

corresponds to the arrangement of dominoes

with the doubles
00,11,22,33,44,55,66

inserted at appropriate places.
Eulerian trail

**Problem** By finding an Eulerian trail in $K_5$, arrange a set of fifteen dominos (from 00 to 44) in a domino ring.
Hamiltonian graphs

The term ‘Hamiltonian’ derives from icosian game, invented by Sir William Rowan Hamilton.

The player has to find Hamiltonian cycles starting with 5 given letters. For example, starting with BCPNM:

- BCPNMDFKLTQRZXWVJHGB
- BCPNMDFGHXWVJLTKRQZB
Hamiltonian graphs

The game was marketed in 1859, also in a solid dodecahedron form under the title *A voyage round the world* with vertices representing places.
Hamiltonian graphs

**Problem** Find a path on the dodecahedron starting with $BCD$, ending with $T$, and including each vertex exactly once.
Properties of Hamiltonian graphs

Unlike for Eulerian graphs, no general necessary and sufficient conditions for Hamiltonicity are known. Some particular classes of graphs are known to be Hamiltonian, for example, $C_n$ and $K_n$.

**Problem** Which of the following graphs are Hamiltonian?
(a) The complete bipartite graph $K_{4,4}$;
(b) A tree.
Properties of Hamiltonian graphs

**Problem**

(a) Prove that a bipartite graph with an odd number of vertices is not Hamiltonian.

(b) Use the result of part (a) to prove that the following graph is not Hamiltonian.
Properties of Hamiltonian graphs

**Theorem (Ore’s Theorem)** Let $G$ be a simple connected graph with $n$ vertices, $n > 3$, and
\[ \text{deg } v + \text{deg } w \geq n \]
for each pair of non-adjacent vertices $v$ and $w$. Then $G$ is Hamiltonian.

Example:
Properties of Hamiltonian graphs

Problem  (a) Let $G$ be a simple connected graph with $n$ vertices, $n > 3$ and $\text{deg } v \geq n/2$ for each vertex $v$. Use Ore’s Theorem to show that $G$ is Hamiltonian. (This result is known as Dirac’s Theorem (1952)).

(b) Give an example of a Hamiltonian graph that does not satisfy the conditions of Ore’s Theorem.
Semi-Hamiltonian graphs

A connected graph is **semi-Hamiltonian** if there is a path that includes every vertex; such a path is a **semi-Hamiltonian path**.

**Problem** Determine which of the following graphs are semi-Hamiltonian, and write down a corresponding semi-Hamiltonian path, where possible.
Knight’s tour problem

On a chessboard, a knight always moves two squares in a horizontal or vertical direction and one square in a perpendicular direction.
Knight’s tour problem

Knight’s tour problem: Can a knight visit each square of a chessboard exactly once by a sequence of knight’s moves, and finish on the same square as it began?

We represent the board as a graph: each vertex corresponds to a square; each edge corresponds to a pair of squares connected by a knight’s move.
Knight’s tour problem

Note that there is no knight’s tour on a 4x4 chessboard.
Knight’s tour problem

Knight’s tour does exist for 8x8 chessboard.
Edge colouring

Example: Wire Colouring

We wish to make a display panel on which electrical components $a,b,\ldots$ are to be mounted and interconnected. The connecting wires are formed into a cable, with the wires to be connected to $a$ emerging through one hole in the panel, those connected to $b$ through another hole, and so on. To distinguish the wires that are coming through the same hole, they are coloured differently. What is the minimum number of colours necessary for the whole system?

We represent the connection points by the vertices of a graph, and the wires by edges.
Edge colouring

Example (cont.) This graph represents a panel with 6 components.

Since vertex b has 5 edges incident with it, all to be coloured differently, at least 5 colours are necessary. We can see that 5 colours is also sufficient.
Chromatic index

Let G be a graph without loops. A \textit{k-edge colouring} of G is an assignment of at most \( k \) colours to the edges of G in such a way that any two edges meeting at a vertex are assigned different colours.

If G has a \( k \)-edge colouring then G is \textit{k-edge colourable}. The \textit{chromatic index} of G, denoted by \( \chi'(G) \), is the smallest number \( k \) for which G is \( k \)-edge colourable.

\textbf{Note} that the definitions are for graphs without loops.
Chromatic index

We usually show a $k$-edge colouring by writing the numbers $1, 2, \ldots, k$ next to the appropriate edges.

*Below are examples of edge colourings.*
Chromatic index

**Problem** Determine $\chi'(G)$ for each of the following graphs $G$.

**Hint.** For each graph, devise a suitable colouring and explain why there is no colouring with fewer colours.
Chromatic index

**Problem** Write down the chromatic index for each of the following graphs.
(a) the complete graph $K_4$;
(b) the complete bipartite graph $K_{2,3}$;
(c) the cycle graph $C_6$.

**Problem** Decide whether each of the following statements about a graph $G$ is true or false, and give a proof or counter-example, as appropriate.
(a) If $G$ contains a vertex of degree $r$ then $\chi'(G) \geq r$.
(b) If $\chi'(G) \geq r$ then $G$ contains a vertex of degree $r$. 
Chromatic index

Given a particular graph $G$, how can we determine its chromatic index?

Upper bound for $\chi'(G)$ can be found by constructing an explicit colouring for the edges of $G$.

Lower bound for $\chi'(G)$ can be found by finding the largest vertex degree of $G$.

For example, this graph will need at least 5 colours since the maximum degree is 5.

Note that if a graph $G$ has $m$ edges then $\chi'(G) \leq m$. Equality holds when $G$ is a complete bipartite graph $K_{1,m}$. 
Chromatic index

**Theorem 13.1 Vizing’s Theorem**
Let $G$ be a simple graph whose maximum vertex degree is $\Delta$. Then

$$\Delta \leq \chi'(G) \leq \Delta + 1.$$  

**Proof** is omitted.

*This means that we can classify all graphs into two classes:*
1. Graphs for which $\chi'(G) = \Delta$; and
2. Graphs for which $\chi'(G) = \Delta + 1$.

**Note** that it is not known in general which graphs belong to which class.
Chromatic index

Problem For each of the following graphs $G$, write down the lower and upper bounds for $\chi'(G)$ given by Vizing’s theorem; the actual value of $\chi'(G)$ and a colouring using $\chi'(G)$ colours.
(a) the cycle graph $C_7$;
(b) the complete bipartite graph $K_{2,4}$;
(c) the complete graph $K_6$. 
Chromatic index

To find the chromatic index $\chi'(G)$ of a simple graph $G$. Try to find an upper bound and a lower bound that are the same; then $\chi'(G)$ is equal to this common value.

Possible upper bounds for $\chi'(G)$:
the number of colours used in an explicit edge colouring of $G$; the number $m$ of edges in $G$; $\Delta + 1$, where $\Delta$ is the maximum vertex degree in $G$, provided that $G$ has no multiple edges (Vizing’s theorem);

Possible lower bounds for $\chi'(G)$:
$\Delta$, the maximum vertex degree in $G$. 
Classifying some simple graphs

We can classify all graphs into two classes:
1. Graphs for which $\chi'(G) \leq \Delta$; and
2. Graphs for which $\chi'(G) \leq \Delta + 1$.

For some families of graphs this is simple.

$\chi'(C_n) = 2,$ if $n$ is even;
$\quad = 3,$ if $n$ is odd.
Classifying some simple graphs

For the complete graph $K_5$ and $K_6$ we have the following edge colourings.
Classifying some simple graphs

More generally, we have

**Theorem** For the complete graph $K_n$,
\[
\chi'(K_n) = \begin{cases} 
  n - 1, & \text{if } n \text{ is even;} \\
  n, & \text{if } n \text{ is odd.}
\end{cases}
\]

**Konig’s Theorem**
Let $G$ be a bipartite graph whose maximum vertex degree is $\Delta$. Then
\[
\chi'(G) = \Delta.
\]
Chromatic index

**Problem** (a) Suppose that 31 teams take part in a competition in which each team must play exactly one match against each of the other 30 teams. If no team can play more than one match a day, how many days are needed?

(b) What is the corresponding answer if there are 32 teams, each of which must play exactly one match against each of the other 31 teams?

**Problem** Use Konig’s theorem to write down the chromatic index of each of the following graphs.
(a) the complete bipartite graph $K_{r,s}$ ($r \leq s$);
(b) the graph of the cube;
(c) the $k$-cube $Q_k$. 

40
Algorithm for edge colouring

**Greedy Algorithm for Edge Colouring**

START with a graph G and a list of colours 1,2,…

Step 1 label the edges $a,b,c,…$ in any manner.

Step 2 identify the uncoloured edge labelled with the earliest letter in the alphabet; colour it with the first colour in the list not used for any coloured edge that meets it at a vertex.

Repeat Step 2 until all edges are coloured, then STOP.

An edge colouring of $G$ is obtained. The number of colours used depends on the labelling chosen for the edges in Step 1.
Algorithm for edge colouring

Illustration A Find an edge colouring of the following graph:

Step 1. We label the edges $a, b, c, d, e, f, g$
Algorithm for edge colouring

Illustration A (cont.) Step 2. We successively colour edge $a$ with colour 1, edge $b$ with colour 2, edge $c$ with colour 1, edge $d$ with colour 3, edge $e$ with colour 3, edge $f$ with colour 2, edge $g$ with colour 4.

All the vertices are now coloured so we STOP.
Algorithm for edge colouring

Illustration B Find an edge colouring of the following graph:

Step 1. We label the edges $a, b, c, d, e, f, g$
Algorithm for edge colouring

Illustration B (cont.) Step 2. We successively colour edge a with colour 1, edge b with colour 1, edge c with colour 2, edge d with colour 2, edge e with colour 3, edge f with colour 3, edge g with colour 3.

All the edges are now coloured so we STOP.

Using the same graph and the same greedy algorithm, we have obtained two different edge colourings.
Algorithm for edge colouring

**Problem** Use the greedy algorithm to colour the edges of the following graph $G$, using each of the given labellings. What is the actual value of $\chi'(G)$?
**Edge decomposition**

Some of the most interesting problems in graph theory involve the decomposition of a graph $G$ into subgraphs of a particular type. In some problems we split the set of edges into disjoint subsets; this is called an *edge decomposition* of $G$.

**Example.** An edge decomposition of the graph below: \{a,b,c\}, \{d,e,f,g,h\} – disjoint subsets that correspond to the components of $G$. 

![Graph](image)
Edge decomposition

Another natural edge decomposition arises from the idea of an Eulerian graph.

Example. For the Eulerian graph $G$ below, there are 5 edge decompositions of $G$ into disjoint cycles.

\{a,b,c,d,e,f\}, \{g,h,i\};
\{a,f,i\}, \{b,c,g\}, \{d,e,h\};
\{a,f,h,g\}, \{b,c,d,e,i\};
\{b,c,h,i\}, \{a,f,e,d,g\};
\{d,e,i,g\}, \{a,b,c,h,f\};
Decomposition into matchings

The following diagram shows the cube graph and 3 sets of edges indicated by thick lines.

These 3 sets have the property that each edge of the graph appears just once – edge decomposition: 
\{ab, cd, ef, gh\}, \{ad, bc, eh, fg\}, \{ac, bf, cg, dh\}.
Each of the 3 sets consists of edges that have no vertex in common. Such a set of edges is called a matching.
Decomposition into matchings

A **matching** in a graph $G$ is a set of edges of $G$, no two of which have a vertex in common.

*Every graph can be decomposed into matchings, since if there are $m$ edges, then we can simply take $m$ matchings, each consisting of a single edge.*

The problem of determining the minimum number of matchings needed to decompose a given graph is unsolved in general.

**Note** that the problem of decomposing a graph into the minimum number of matchings is an edge colouring problem in which the edges of each matching are assigned the same colour.
Scheduling examinations

Example: Scheduling Examinations
How many examination periods are required to set examinations?

Take simple example with 4 students $a,b,c,d$. We represent the students and tutors by the vertices of a bipartite graph and join a student vertex to a tutor vertex whenever the student needs to be examined by the tutor.

If 2 edges meet at a common vertex then the corresponding examination cannot take place simultaneously. So the problem is an edge decomposition problem, a matching.
Scheduling examinations

Example: Scheduling Examinations (cont.)
In this example the minimum number of matchings is 3:

The corresponding edge decomposition is \{aA,bB,dC\}, \{aC,bA,cB\}, \{bC,cA,dB\}.
Scheduling examinations

Problem: Five students $a, b, c, d, e$ are to be examined by five tutors $A, B, C, D, E$:

- Tutor A must examine students $b$ and $d$;
- Tutor B must examine students $a, b$ and $e$;
- Tutor C must examine students $b, c$ and $e$;
- Tutor D must examine students $a$ and $c$;
- Tutor A must examine students $b, d$ and $e$.

If each examination takes the same amount of time, find the minimum number of examination periods needed and devise a suitable schedule.
Revision (and terms to know)

- Eulerian and Hamiltonian Graphs
- Semi-Eulerian and Semi-Hamiltonian Graphs
- Edge colouring
- Edge chromatic number
- Greedy algorithm
- Edge decomposition
- Scheduling