Session 1: Graceful Labelings

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Extremal Problems and Hamiltonicity in Graphs
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Overview

1. Graceful Labelings and edge decompositions

2. Some examples of graceful graphs

3. Some general results on gracefulness

4. Conjectures and open problems
Graceful Labelings

Notation: Graph $G = (V, E)$ with $n$ vertices and $m$ edges.

Definition

A graceful labeling of $G$ is an injective map

$$f : V \rightarrow \{0, 1, \ldots, m\}$$

such that the induced edge values

$$f_e : E \rightarrow \mathbb{N}$$

$$xy \mapsto |f(x) - f(y)|$$

are pairwise distinct.
An edge decomposition of a graph $H$ is a partition of its edge set $E(H)$. We say that $G$ decomposes $H$, and write $G\mid H$, if $H$ admits an edge decomposition into copies of $G$.

$P_2\mid K_5$. 

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Proposition

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**Lemma**

If there is a map $f : V \rightarrow [0, k]$ such that the induced edge values $\{|f(x) - f(y)| : xy \in E(G)\}$ are pairwise distinct then there is a $2m$–regular graph $H$ such that $G$ decomposes $H$.

**Theorem (Wilson, 1975)**

For every graph $G$ with no isolated vertices there is $k$ such that $G$ decomposes $K_{2k+1}$.

**Definition**

The minimum $k$ for which there is a map $f : V \rightarrow [0, k]$ with pairwise distinct edge values is the **gracesize** $gr(G)$ of $G$.
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- The grid $P_r \times P_s$ is graceful for all $r, s \geq 1$. Maheo, 1980
- The torus $C_r \times C_s$ is graceful if $r \equiv 0 \pmod{4}$ and $s$ even, it is not if $m$ and $n$ both odd. Jungreis, Reid, 1992
- The wheel $W_n$ is graceful for all $n \geq 4$. Frucht, 1977
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Tree with at most four ends. Huang, Kotzig, Rosa, 1982
Tree with at most 27 vertices. Aldred, McKay, 2007
Some general results

...on graceful graphs

- If $G$ is Eulerian and graceful then $m \equiv 0, 3 \pmod{4}$ (parity condition).
  
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- Almost all graphs are not graceful.
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- Every subdivision of a graceful tree is graceful.
  Burzio and Ferrarese (1998)

- Every tree of diameter at most five is graceful.
  Hrnˇ ciar and Haviar (2001)

- Every rooted tree with all vertices of even degree is graceful.
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- The gracesize of a tree with \( m \) edges is at most \( 5m/3 \).
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- Every tree with \( m \) edges is a subtree of a graceful tree with at most \( 3m/2 \) edges.
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