LAGUERRE SEMIGROUP AND DUNKL OPERATORS

SALEM BEN SAÏD, TOSHIYUKI KOBAYASHI, AND BENT ØRSTED

1. Introduction

1.1. Holomorphic semigroup $\mathscr{I}_{k,a}(z)$ with two parameters k and a.

operators. To fix notation, let $\mathfrak{C}$ be the Coxeter group associated with a root system $\mathscr{R}$ in $\mathbb{R}^N$. For a $\mathfrak{C}$-invariant function $k \equiv (k_\alpha)$ (*multiplicity function*) on $\mathscr{R}$, we write Δ_k for the Dunkl Laplacian on $\mathbb{R}^N$

We take $a > 0$ to be a deformation parameter, and introduce the following differential-difference operator

$$\Delta_{k,a} := \|x\|^{2-a}\Delta_k - \|x\|^a. \qquad (1.1)$$

Here, $\|x\|$ is the norm of the coordinate $x \in \mathbb{R}^N$, and $\|x\|^a$ in the right-hand side of the formula stands for the multiplication operator by $\|x\|^a$. Then, $\Delta_{k,a}$ is a symmetric operator on the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ consisting of square integrable functions on $\mathbb{R}^N$ against the measure $\vartheta_{k,a}(x)dx$, where the density $\vartheta_{k,a}(x)$ is given by

$$\vartheta_{k,a}(x) := \|x\|^{a-2} \prod_{\alpha \in \mathscr{R}} |\langle \alpha, x \rangle|^{k_\alpha}. \tag{1.2}$$

The (k, a)*-generalized Laguerre semigroup* $\mathscr{I}_{k,a}(z)$ is defined to be the semigroup with infinitesimal generator $\frac{1}{a}\Delta_{k,a}$, that is,

$$\mathscr{I}_{k,a}(z) := \exp\Big(\frac{z}{a}\Delta_{k,a}\Big), \tag{1.3}$$

for $z \in \mathbb{C}$ such that $\operatorname{Re} z \geq 0$. (Later, we shall use the notation $\mathscr{I}_{k,a}(z) = \Omega_{k,a}(\gamma_z)$, in connection with the Gelfand–Gindikin program.)

In the case $a = 2$ and $k = 0$, we have $\vartheta_{0,2}(x) \equiv 1$ and recover the classical setting where

We shall prove

Theorem A.

1) $\Delta_{k,a}$ *extends to a self-adjoint operator on* $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.
2) *There is no continuous spectrum of* $\Delta_{k,a}$ *for any* $a > 0$ *and for any non-negative multiplicity-function k.*

We also find all the discrete spectra explicitly.

Turning to the (k, a)-generalized Laguerre semigroup $\mathscr{I}_{k,a}(z)$, we prove:

Theorem B.

1) $\mathscr{I}_{k,a}(z)$ *is a holomorphic semigroup in the complex right-half plane* $\{z \in \mathbb{C} : \mathrm{Re}\, z >$

$0\}$ *in the sense that* $\mathscr{I}_{k,a}(z)$ *is a Hilbert–Schmidt operator on* $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ *satisfying*

$$\mathscr{I}_{k,a}(z_1)\circ\mathscr{I}_{k,a}(z_2) = \mathscr{I}_{k,a}(z_1+z_2), \quad (\operatorname{Re} z_j > 0),$$

and that the scalar product $(\mathscr{I}_{k,a}(z)f, g)$ *is a holomorphic function of* z *for* $\operatorname{Re} z > 0$, *for any* $f, g \in L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.

2) $\mathscr{I}_{k,a}(z)$ *is a one-parameter group of unitary operators on the imaginary axis* $\operatorname{Re} z = 0$.

For the particular values $a = 1$ and 2, we find a closed formula for the density of the semigroup $\mathscr{I}_{k,a}(z)$ by means of the Dunkl intertwining operator V_k and Bessel functions.

Theorem C. *Suppose* $a = 1$ *or* 2. *Then, for* $z \in \mathbb{C}$ *such that* $\operatorname{Re} z > 0$,

$$(\exp(z\Delta_{k,a})f)(x) = c_{k,a} \int_{\mathbb{R}^N} K_{k,a}(x, y; az) f(y)\vartheta_{k,a}(y)dy,$$

where $c_{k,a}$ is a constant obtained by Opdam as a solution to a conjecture of Macdonald, and the integral kernel is expressed in terms of Bessel functions.

1.2. (k,a)-generalized Fourier transforms $\mathscr{F}_{k,a}$.

As we mentioned in Theorem B (2), the 'boundary value' of the (k,a)-generalized Laguerre semigroup $\mathscr{I}_{k,a}(z)$ on the imaginary axis gives a one-parameter family of unitary operators. The case $z = 0$ gives the identity operator, namely, $\mathscr{I}_{k,a}(0) = \mathrm{id}$. The particularly interesting case is when $z = \frac{\pi i}{2}$, and we set

$$\mathscr{F}_{k,a} := c\, \mathscr{I}_{k,a}\Big(\frac{\pi i}{2}\Big) = c \exp\Big(\frac{\pi i}{2a}(\|x\|^{2-a}\Delta_k - \|x\|^a)\Big)$$

by multiplying the phase factor $c = e^{i\frac{\pi}{2}(\frac{2\langle k\rangle+N+a-2}{a})}$ (see (5.2)). Then, the unitary operator $\mathscr{F}_{k,a}$ for general a and k satisfies the following significant properties:

Theorem D. *For any $a > 0$ and any non-negative multiplicity function k, $\mathscr{F}_{k,a}$ is a unitary operator on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ satisfying the following formulas for any $f \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$.*

1) $\mathscr{F}_{k,a}(Ef) = -(E+N+2\langle k\rangle+a-2)(\mathscr{F}_{k,a}f)$.
2) $\mathscr{F}_{k,a}(\|x\|^a f) = -\|x\|^{2-a}\Delta_k(\mathscr{F}_{k,a}f)$.
3) $\mathscr{F}_{k,a}(\|x\|^{2-a}\Delta_k f) = -\|x\|^a(\mathscr{F}_{k,a}f)$.
4) $\mathscr{F}_{k,a}^{2p} = \mathrm{id}$ *if a is a rational number of the form $a = \frac{p}{q}$.*

We call $\mathscr{F}_{k,a}$ a (k,a)*-generalized Fourier transform* on $\mathbb{R}^N$. We note that $\mathscr{F}_{k,a}$ reduces to the Euclidean Fourier transform $\mathscr{F}$ if $k \equiv 0$ and $a = 2$; to the Hankel transform if $k \equiv 0$ and $a = 1$; to the Dunkl transform $\mathscr{D}_k$ introduced by C. Dunkl himself if $k > 0$ and $a = 2$.

Thus, in these classical setting, our approach uses the following expressions of $\mathscr{F}_{k,a}$:

$$\mathscr{F} = e^{\frac{\pi i N}{4}} \exp \frac{\pi i}{4}(\Delta - \|x\|^2), \qquad \text{(Fourier)}$$

$$\mathscr{D}_k = e^{\frac{\pi i (2\langle k\rangle + N)}{4}} \exp \frac{\pi i}{4}(\Delta_k - \|x\|^2). \qquad \text{(Dunkl)}$$

For $a = 1$ and $k \equiv 0$, the unitary operator

$$\mathscr{F}_{0,1} = e^{\frac{\pi i (N-1)}{2}} \exp\Big(\frac{\pi i}{2}\|x\|(\Delta - 1)\Big)$$

arises as the *unitary inversion operator* of the Schrödinger model of the minimal representation of the conformal group $O(N+1,2)$. Its Dunkl analogue, namely, the unitary operator $\mathscr{F}_{k,a}$ for $a = 1$ and $k > 0$ seems also interesting. Thus, we set

$$\mathscr{H}_k := \mathscr{F}_{k,1} = e^{i\frac{\pi}{2}(2\langle k\rangle + N - 1)} \mathscr{I}_{k,1}\Big(\frac{\pi i}{2}\Big).$$

For a general (k,a), we find the inversion formula, the Plancherel theorem, the Hecke identity, the Bochner identity, and the Heisenberg inequality for the (k,a)-generalized Fourier transform $\mathscr{F}_{k,a}$. As for the Heisenberg type inequality, we note that such an inequality was previously proved by Rösler and Shimeno for the $a = 2$ case (i.e. the Dunkl transform $\mathscr{D}_k$).

1.3. $\mathfrak{sl}_2$-triple of differential-difference operators.

The basic tool for the presently is the SL_2 theory. The key idea is to construct an $\mathfrak{sl}_2$-triple of differential-difference operators with two parameters k and a. We then apply representation theory of $\widetilde{SL(2,\mathbb{R})}$, the universal covering group of $SL(2,\mathbb{R})$. The resulting representation is a discretely decomposable unitary representation, which depends continuously on parameters a and k.

To be more precise, we introduce the following differential-difference operators on $\mathbb{R}^N \setminus$

$\{0\}$ by

$$\mathbb{E}^+_{k,a} := \frac{i}{a}\|x\|^a, \qquad \mathbb{E}^-_{k,a} := \frac{i}{a}\|x\|^{2-a}\Delta_k,$$

$$\mathbb{H}_{k,a} := \frac{2}{a}\sum_{i=1}^{N} x_i\partial_i + \frac{N+2\langle k\rangle + a - 2}{a}.$$

With these operators, we have

$$a\Delta_{k,a} = i\,(\mathbb{E}^+_{k,a} - \mathbb{E}^-_{k,a}).$$

The main point here is that our operator $\Delta_{k,a}$ can be interpreted in the framework of the (infinite dimensional) representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$:

Lemma E. *The differential-difference operators* $\{\mathbb{H}_{k,a}, \mathbb{E}^+_{k,a}, \mathbb{E}^-_{k,a}\}$ *form an* $\mathfrak{sl}_2$*-triple for any multiplicity-function* k *and any non-zero complex number* a.

In other words, taking a basis of $\mathfrak{sl}(2,\mathbb{R})$ as

$$\mathbf{e}^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{e}^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

we get a Lie algebra representation $\omega_{k,a}$ of $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ with continuous parameters k and a on functions on $\mathbb{R}^N$ by mapping

$$\mathbf{h} \mapsto \mathbb{H}_{k,a}, \quad \mathbf{e}^+ \mapsto \mathbb{E}^+_{k,a}, \quad \mathbf{e}^- \mapsto \mathbb{E}^-_{k,a}.$$

The main result is to prove that the representation $\omega_{k,\ell}$ of $\mathfrak{sl}(2,\mathbb{R})$ lifts to the universal covering group $\widetilde{SL(2,\mathbb{R})}$:

Theorem F. *If $a > 0$ and k is non-negative, then $\omega_{k,a}$ lifts to a unitary representation of $\widetilde{SL(2,\mathbb{R})}$ on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.*

The Hilbert space $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ decomposes discretely as a direct sum of unitary representations of the direct product group $\mathfrak{C} \times \widetilde{SL(2,\mathbb{R})}$:

$$L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx) \simeq$$

$$\sum_{m=0}^{\infty}{}^{\oplus} \mathscr{H}_k^m(\mathbb{R}^N) \otimes \pi\Big(\frac{2m + 2\langle k\rangle + N - 2}{a}\Big),$$

where $\mathscr{H}_k^m(\mathbb{R}^N)$ stands for the representation of the Coxeter group $\mathfrak{C}$ on the eigenspace of the Dunkl Laplacian (the space of spherical k-harmonics of degree m) and $\pi(\nu)$ is an irreducible unitary lowest weight representation of $S\widetilde{L(2,\mathbb{R})}$ of weight $\nu + 1$. The unitary isomorphism is constructed explicitly by using Laguerre polynomials.

The unitary representation of $S\widetilde{L(2,\mathbb{R})}$ on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ extends furthermore to a holomorphic semigroup of a complex three dimensional semigroup (see Section 3.4). Basic properties of the holomorphic semigroup $\mathscr{I}_{k,a}(z)$ defined in (1.3) and the unitary operator $\mathscr{F}_{k,a}$ can be read from the 'dictionary' of $\mathfrak{sl}(2, \mathbb{R})$ as follows:

$$i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \longleftrightarrow \frac{1}{a}\Delta_{k,a}$$

$$\exp iz\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \longleftrightarrow \mathscr{I}_{k,a}(z) = \exp(\frac{z}{a}\Delta_{k,a})$$

$$w_0 = \exp\frac{\pi}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \longleftrightarrow \mathscr{F}_{k,a} = c\mathscr{I}_{k,a}\Big(\frac{\pi i}{2}\Big)$$

$$\mathbf{e}^{-} \longleftrightarrow \mathscr{F}_{k,a} \circ \|x\|^{a}$$

$$\mathbf{e}^{+} \longleftrightarrow \mathscr{F}_{k,a} \circ \|x\|^{2-a}\Delta_{k}.$$

1.4. **Hidden symmetries for** $a = 1$ **and** 2.

As we have seen in Section 1.1, one of the reasons that we find an explicit formula for the holomorphic semigroup $\mathscr{I}_{k,a}(z)$ (and for the unitary operator $\mathscr{F}_{k,a}$) (see Section 1.1) is that there are large 'hidden symmetries' on the Hilbert space when $a = 1$ or 2.

We recall that our analysis is based on the fact that the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ has a symmetry of the direct product group $\mathfrak{C} \times \widetilde{SL(2, \mathbb{R})}$ for all k and a. It turns out that this symmetry becomes larger for special values of k and a. In this subsection, we discuss these hidden symmetries.

First, in the case $k \equiv 0$, the Dunkl-Laplacian Δ_k becomes the Euclidean Laplacian Δ, and consequently, not only the Coxeter group $\mathfrak{C}$ but also the whole orthogonal group $O(N)$ commutes with $\Delta_k \equiv \Delta$. Therefore, the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{0,a}(x)dx)$ is acted on by $O(N)\times \widetilde{SL(2,\mathbb{R})}$. Namely, it has a larger symmetry

$$\mathfrak{C} \times \widetilde{SL(2,\mathbb{R})} \subset O(N) \times \widetilde{SL(2,\mathbb{R})}.$$

Next, we observe that the Lie algebra of the direct product group $O(N)\times \widetilde{SL(2,\mathbb{R})}$ may be seen as a subalgebra of two different reductive Lie algebras $\mathfrak{sp}(N,\mathbb{R})$ and $\mathfrak{o}(N+1,2)$:

$$\mathfrak{o}(N) \oplus \mathfrak{sl}(2,\mathbb{R}) \simeq \mathfrak{o}(N) \oplus \mathfrak{o}(1,2) \subset \mathfrak{o}(N+1,2)$$
$$\mathfrak{o}(N) \oplus \mathfrak{sl}(2,\mathbb{R}) \simeq \mathfrak{o}(N) \oplus \mathfrak{sp}(1,\mathbb{R}) \subset \mathfrak{sp}(N,\mathbb{R})$$

It turns out that they are the hidden symmetries of the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{0,a}(x)dx)$

for $a = 1, 2$. To be more precise, the conformal group $O(N+1,2)_0$ (or its double covering group if N is even) acts on $L^2(\mathbb{R}^N, \vartheta_{0,1}(x)dx) = L^2(\mathbb{R}^N, \|x\|^{-1}dx)$ as an irreducible unitary representation, while the metaplectic group $Mp(N,\mathbb{R})$ (the double covering group of the symplectic group $Sp(N,\mathbb{R})$) acts on $L^2(\mathbb{R}^N, \vartheta_{0,2}(x)dx) = L^2(\mathbb{R}^N, dx)$ as a unitary representation.

2. Preliminary results on Dunkl operators

2.1. **Dunkl operators.**

Let $\langle \cdot, \cdot \rangle$ be the standard Euclidean scalar product in $\mathbb{R}^N$. We shall use the same notation for its bilinear extension to $\mathbb{C}^N \times \mathbb{C}^N$. For $x \in \mathbb{R}^N$, denote by $\|x\| = \langle x, x\rangle^{1/2}$

For $\alpha \in \mathbb{R}^N \setminus \{0\}$, let r_α be the reflection in the hyperplane $\langle \alpha \rangle^\perp$ orthogonal to α

$$r_\alpha(x) := x - 2\frac{\langle \alpha, x\rangle}{\|\alpha\|^2}\alpha, \qquad x \in \mathbb{R}^N.$$

We say a finite set $\mathscr{R} \subset \mathbb{R}^N$ of non-zero vectors is a reduced root system if:

(R1) $r_\alpha(\mathscr{R}) = \mathscr{R}$ for all $\alpha \in \mathscr{R}$,
(R2) $\mathscr{R} \cap \mathbb{R}\alpha = \{\pm\alpha\}$ for all $\alpha \in \mathscr{R}$

In this paper, we do not impose crystallographic conditions on the roots, and do not require that $\mathscr{R}$ spans $\mathbb{R}^N$. Each root system can be written as a disjoint union $\mathscr{R} = \mathscr{R}^+ \cup (-\mathscr{R}^+)$, where $\mathscr{R}^+$ and $(-\mathscr{R}^+)$ are separated by a hyperplane through the origin. The set $\mathscr{R}^+$ is called a positive subsystem and its choice is not unique.

The subgroup $\mathfrak{C} \subset O(N, \mathbb{R})$ generated by the reflections $\{r_\alpha \mid \alpha \in \mathscr{R}\}$ is called the finite Coxeter group associated with $\mathscr{R}$. The Weyl groups such as the symmetric group $\mathfrak{S}_N$ for the type A_{N-1} root system and the hyperoctahedral group for the type B_N root system are the case. In addition, H_3, H_4 (icosahedral groups) and $I_2(n)$ (symmetry group of the regular n-gon) are the Coxeter groups.

Definition 2.1. *A multiplicity function for $\mathfrak{C}$ is a function $k : \mathscr{R} \to \mathbb{C}$ which is constant on $\mathfrak{C}$-orbits.*

Setting $k_\alpha := k(\alpha)$ for $\alpha \in \mathscr{R}$, we have $k_{h\alpha} = k_\alpha$ for all $h \in \mathfrak{C}$ from definition. We say k is non-negative if $k_\alpha \geq 0$ for all $\alpha \in \mathscr{R}$. The $\mathbb{C}$-vector space of multiplicity functions on $\mathscr{R}$ is denoted by $\mathscr{K}$. The dimension of $\mathscr{K}$ is equal to the number of $\mathfrak{C}$-orbits in $\mathscr{R}$.

For $\xi \in \mathbb{C}^N$ and $k \in \mathscr{K}$, C. Dunkl introduced a family of first order differential-difference operators $T_\xi(k)$ (*Dunkl's operators*) by

$$T_\xi(k)f(x) := \partial_\xi f(x) + \sum_{\alpha \in \mathscr{R}^+} k_\alpha \langle \alpha, \xi \rangle \frac{f(x) - f(r_\alpha x)}{\langle \alpha, x \rangle}. \tag{2.1}$$

Here ∂_ξ denotes the directional derivative corresponding to ξ. Thanks to the $\mathfrak{C}$-invariance of the multiplicity function, this definition is independent of the choice of the positive subsystem $\mathscr{R}^+$. The operators $T_\xi(k)$ are homogeneous of degree -1. Moreover the Dunkl operators satisfy:

(D1) $L(h) \circ T_\xi(k) \circ L(h)^{-1} = T_{h\xi}(k)$ for all $h \in \mathfrak{C}$,

(D2) $T_\xi(k)T_\eta(k) = T_\eta(k)T_\xi(k)$ for all $\xi, \eta \in \mathbb{R}^N$,

(D3) $T_\xi(k)[fg] = gT_\xi(k)f + fT_\xi(k)g$ if f and g are in $C^1(\mathbb{R}^N)$ and at least one of them is $\mathfrak{C}$-invariant.

Here, we denote by $L(h)$ the left regular action of $h \in \mathfrak{C}$ on the function space on $\mathbb{R}^N$:

$$(L(h)f)(x) := f(h^{-1} \cdot x).$$

Let ϑ_k be the weight function on $\mathbb{R}^N$ defined by

$$\vartheta_k(x) := \prod_{\alpha \in \mathscr{R}^+} |\langle \alpha, x \rangle|^{2k_\alpha}, \qquad x \in \mathbb{R}^N. \tag{2.2}$$

It is $\mathfrak{C}$-invariant and homogeneous of degree $2\langle k\rangle$, where the index of the multiplicity function k is defined as

$$\langle k\rangle := \sum_{\alpha\in\mathscr{R}^+} k_\alpha = \frac{1}{2}\sum_{\alpha\in\mathscr{R}} k_\alpha. \qquad (2.3)$$

We consider for non-negative root multiplicity functions the unique linear isomorphism V_k (*Dunkl's intertwining operator*) on the space $\mathscr{P}(\mathbb{R}^N)$ of polynomial functions on $\mathbb{R}^N$ such that

(I1) $V_k(\mathscr{P}_m(\mathbb{R}^N)) = \mathscr{P}_m(\mathbb{R}^N)$ for all $m \in \mathbb{N}$,
(I2) $V_{k|\mathscr{P}_0(\mathbb{R}^N)} = \mathrm{id}$,
(I3) $T_\xi(k)V_k = V_k\partial_\xi$ for all $\xi \in \mathbb{R}^N$.

The intertwining operator V_k plays a fundamental role in Dunkl's theory.

For arbitrary finite reflection group $\mathfrak{C}$, and for any non-negative multiplicity function k, Rösler [**?**] proved that there exists a unique positive Radon probability-measure μ_x^k on $\mathbb{R}^N$

such that

$$V_k f(x) = \int_{\mathbb{R}^N} f(\xi) d\mu_x^k(\xi). \tag{2.4}$$

The support of μ_x^k is contained in the ball $\{\xi \in \mathbb{R}^N \mid \|\xi\| \leq \|x\|\}$. Moreover, for any Borel set $S \subset \mathbb{R}^N$, $g \in \mathfrak{C}$ and $r > 0$, the following invariant property holds:

$$\mu_x^k(S) = \mu_{gx}^k(gS) = \mu_{rx}^k(rS).$$

2.2. **Dunkl Laplacian.**

The Dunkl–Laplace operator, or simply, the Dunkl Laplacian, is defined as

$$\Delta_k := \sum_{j=1}^{N} T_{\xi_j}(k)^2, \tag{2.5}$$

where $\{\xi_1, \ldots, \xi_N\}$ is an arbitrary orthonormal basis of $(\mathbb{R}^N, \langle\cdot,\cdot\rangle)$. If $k \equiv 0$ then the Dunkl–Laplace operator Δ_k equals the Euclidean Laplacian Δ. We can rewrite Δ_k as

$$\Delta_k f(x) = \Delta f(x) +$$

$$\sum_{\alpha\in\mathscr{R}^+} k_\alpha \left\{\frac{2\langle\nabla f(x),\alpha\rangle}{\langle\alpha,x\rangle} - \|\alpha\|^2\frac{f(x)-f(r_\alpha x)}{\langle\alpha,x\rangle^2}\right\},$$

where ∇ denotes the usual gradient operator. Δ_k commutes with the $\mathfrak{C}$-action, i.e.

$$L(h)\circ\Delta_k\circ L(h)^{-1} = \Delta_k, \qquad \forall h\in\mathfrak{C}. \quad (2.6)$$

Definition 2.2. *A k-harmonic polynomial of degree m* ($m\in\mathbb{N}$) *is a homogeneous polynomial p on $\mathbb{R}^N$ of degree m such that $\Delta_k p = 0$.*

Denote by $\mathscr{H}_k^m(\mathbb{R}^N)$ the space of k-harmonic polynomials of degree m, and define the following inner product

$$\langle f,g\rangle_k := d_k\int_{S^{N-1}} f(\omega)g(\omega)\vartheta_k(\omega)d\sigma(\omega),$$

where ϑ_k is the density given in (2.2), and

$$d_k := \Big(\int_{S^{N-1}}\vartheta_k(\omega)d\sigma(\omega)\Big)^{-1}. \qquad (2.7)$$

For $k \equiv 0$, d_k^{-1} is the volume of the unit sphere, namely,

$$d_0 = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}}.$$

As in the $k \equiv 0$ case, we have

Fact 2.3.

1) $\mathscr{H}_k^m(\mathbb{R}^N)|_{S^{N-1}}$ $(m = 0, 1, 2, \dots)$ *is orthogonal to each other with respect to* $\langle\ ,\ \rangle_k$.

2) *The Hilbert space* $L^2(S^{N-1}, \vartheta_k(\omega)d\sigma(\omega))$ *decomposes as a direct Hilbert sum:*

$$L^2(S^{N-1}, \vartheta_k(\omega)d\sigma(\omega)) = \sum_{m\in\mathbb{N}}^{\oplus} \mathscr{H}_k^m(\mathbb{R}^N)|_{S^{N-1}}. \tag{2.8}$$

To end this section, we give a generalization of the classical formula

$$e^{\|x\|^2} \circ \Delta \circ e^{-\|x\|^2} = \Delta + 4\|x\|^2 - 2N - 4\sum_{j=1}^{N} x_j \partial_j$$

to the Dunkl setting.

Lemma 2.4. *For any $\nu \in \mathbb{C}$ and $a \neq 0$, we have*

$$e^{\frac{\nu}{a}||x||^a} \circ ||x||^{2-a}\Delta_k \circ e^{-\frac{\nu}{a}||x||^a} = ||x||^{2-a}\Delta_k + \nu^2||x||^a - \nu\Big((N + 2\langle k\rangle + a - 2) + 2\sum_{j=1}^{N} x_j\partial_j\Big)$$

3. The infinitesimal representation $\omega_{k,a}$ of $\mathfrak{sl}(2,\mathbb{R})$

3.1. $\mathfrak{sl}_2$ triple of differential-difference operators.

In this subsection, we construct a family of Lie algebras which are isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ in the space of differential-difference operators on $\mathbb{R}^N$. This family is parametrized by a non-zero complex number a and a multiplicity function k for the Coxeter group.

We take a basis for the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ as

$$\mathbf{e}^+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{e}^- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{h} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.1}$$

The triple $\{\mathbf{e}^+, \mathbf{e}^-, \mathbf{h}\}$ satisfies the commutation relations

$$[\mathbf{e}^+, \mathbf{e}^-] = \mathbf{h}, \qquad [\mathbf{h}, \mathbf{e}^+] = 2\mathbf{e}^+, \qquad [\mathbf{h}, \mathbf{e}^-] = -2\mathbf{e}^-. \tag{3.2}$$

Definition 3.1. *An $\mathfrak{sl}_2$ triple is a triple of non-zero elements in a Lie algebra satisfying the same relation with* (3.2).

We recall from Section 2 that Δ_k is the Dunkl–Laplacian associated with a multiplicity function k for a Coxeter group $\mathfrak{C}$, and that $\langle k \rangle$ is the index defined in (2.3). For a non-zero complex parameter a, we introduce as before the following differential-difference operators on $\mathbb{R}^N$:

$$\mathbb{E}^+_{k,a} := \frac{i}{a}\|x\|^a, \qquad \mathbb{E}^-_{k,a} := \frac{i}{a}\|x\|^{2-a}\Delta_k,$$

$$\mathbb{H}_{k,a} := \frac{N + 2\langle k\rangle + a - 2}{a} + \frac{2}{a}\sum_{i=1}^{N} x_i\partial_i.$$

The point of the definition is:

Theorem 3.2. *The operators $\mathbb{E}^+_{k,a}$, $\mathbb{E}^-_{k,a}$ and $\mathbb{H}_{k,a}$ form an $\mathfrak{sl}_2$ triple for any complex number $a \neq 0$ and any multiplicity function k.*

These differential-difference operators stabilize $C^\infty(\mathbb{R}^N \setminus \{0\})$, the space of (complex valued) smooth functions on $\mathbb{R}^N \setminus \{0\}$. Thus, for each non-zero complex number a and each multiplicity function k for the Coxeter group, we can define an $\mathbb{R}$-linear map

$$\omega_{k,a} : \mathfrak{sl}(2,\mathbb{R}) \to \mathrm{End}(C^\infty(\mathbb{R}^N \setminus \{0\})) \quad (3.3)$$

by setting

$$\omega_{k,a}(\mathbf{h}) = \mathbb{H}_{k,a}, \quad \omega_{k,a}(\mathbf{e}^+) = \mathbb{E}^+_{k,a},$$

$$\omega_{k,a}(\mathbf{e}^-) = \mathbb{E}^-_{k,a}.$$

Then, Theorem 3.2 implies that $\omega_{k,a}$ is a Lie algebra homomorphism.

We denote by $U(\mathfrak{sl}(2,\mathbb{C}))$ the universal enveloping algebra of the complex Lie algebra $\mathfrak{sl}(2,\mathbb{C}) \simeq \mathfrak{sl}(2,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$. Then, we can extend (3.3) to a $\mathbb{C}$-algebra homomorphism (by the same letter)

$$\omega_{k,a} : U(\mathfrak{sl}(2,\mathbb{C})) \to \mathrm{End}(C^\infty(\mathbb{R}^N \setminus \{0\})).$$

We use the letter L to denote by the left regular representation of the Coxeter group $\mathfrak{C}$ on $C^\infty(\mathbb{R}^N \setminus \{0\})$.

Lemma 3.3. *The two actions L of the Coxeter group $\mathfrak{C}$ and $\omega_{k,a}$ of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ commute.*

It is also convenient to consider the Caley transform

$$\{\mathbf{k}, \mathbf{n}^+, \mathbf{n}^-\} := c\{\mathbf{h}, \mathbf{e}^+, \mathbf{e}^-\}c^{-1}, \tag{3.4}$$

where c is the unitary matrix

$$c = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}.$$

That is,

$$\mathbf{k} = i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\,(\mathbf{e}^+ - \mathbf{e}^-), \tag{3.5 a}$$

$$\mathbf{n}^+ = \frac{1}{2}\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} = \frac{1}{2}(i\,\mathbf{h} + \mathbf{e}^+ + \mathbf{e}^-), \tag{3.5 b}$$

$$\mathbf{n}^- = \frac{1}{2}\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix} = \frac{1}{2}(-i\,\mathbf{h} + \mathbf{e}^+ + \mathbf{e}^-). \tag{3.5 c}$$

Then $\{\mathbf{k}, \mathbf{n}^+, \mathbf{n}^-\}$ forms an $\mathfrak{sl}_2$ triple, and its linear span gives the Lie algebra

$$\mathfrak{su}(1,1) := \{X \in \mathfrak{sl}(2,\mathbb{C}) : X^*\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}X = 0\},$$

which is another real form of $\mathfrak{sl}(2,\mathbb{C})$.

Correspondingly to the right-hand side of (3.5 a – c), the Caley transform of the operators (**??**) amounts to:

$$\widetilde{\mathbb{H}}_{k,a} := \omega_{k,a}(\mathbf{k}) = \frac{\|x\|^a - \|x\|^{2-a}\Delta_k}{a} = -\frac{1}{a}\Delta_{k,a}, \tag{3.6 a}$$

$$\widetilde{\mathbb{E}}^+_{k,a} := \omega_{k,a}(\mathbf{n}^+), \tag{3.6 b}$$

$$\widetilde{\mathbb{E}}^-_{k,a} := \omega_{k,a}(\mathbf{n}^-). \tag{3.6 c}$$

We denote

$$E := \sum_{i=1}^N x_i \partial_i$$

the Euler operator. Clearly, $\{\widetilde{\mathbb{E}}^+_{k,a}, \widetilde{\mathbb{E}}^-_{k,a}, \widetilde{\mathbb{H}}_{k,a}\}$ also forms an $\mathfrak{sl}_2$ triple of differential-difference operators. We shall find another expression of the operators $\widetilde{\mathbb{E}}^+_{k,a}, \widetilde{\mathbb{E}}^-_{k,a}, \widetilde{\mathbb{H}}_{k,a}$ in Lemma 3.4. For this, we note

$$e^{\nu\frac{\|x\|^a}{a}} \circ \|x\|^{2-a}\Delta_k \circ e^{-\nu\frac{\|x\|^a}{a}}$$
$$= \|x\|^{2-a}\Delta_k + \nu^2\|x\|^a - a\nu\mathbb{H}_{k,a}.$$

Here, $\mathbb{H}_{k,a}$ is the first order differential operator defined above.

We are ready to give a second expression of the triple $\{\widetilde{\mathbb{E}}^+_{k,a}, \widetilde{\mathbb{E}}^-_{k,a}, \widetilde{\mathbb{H}}_{k,a}\}$ as follows:

Lemma 3.4. *Let* $\widetilde{\mathbb{E}}^+_{k,a}$, $\widetilde{\mathbb{E}}^-_{k,a}$, *and* $\widetilde{\mathbb{H}}_{k,a}$ *be as in* (3.5 a, b, c). *Then, we have:*

$$\widetilde{\mathbb{E}}^+_{k,a} = \omega_{k,a}(\mathbf{n}^+) = \frac{i}{2a} e^{\frac{\|x\|^a}{a}} \circ \|x\|^{2-a} \Delta_k \circ e^{-\frac{\|x\|^a}{a}}, \tag{3.7 a}$$

$$\widetilde{\mathbb{E}}^-_{k,a} = \omega_{k,a}(\mathbf{n}^-) = \frac{i}{2a} e^{-\frac{\|x\|^a}{a}} \circ \|x\|^{2-a} \Delta_k \circ e^{\frac{\|x\|^a}{a}}, \tag{3.7 b}$$

$$\widetilde{\mathbb{H}}_{k,a} = \omega_{k,a}(\mathbf{k}) = e^{-\frac{\|x\|^a}{a}} \circ \left(\mathbb{H}_{k,a} - \frac{\|x\|^{2-a} \Delta_k}{a} \right) \circ e^{\frac{\|x\|^a}{a}} \tag{3.7 c}$$

3.2. **Laguerre functions revisited.**

In this subsection, we recall the (classical) Laguerre polynomials and give its representation by means of the one parameter group with infinitesimal generator $t\frac{d^2}{dt^2} + (\lambda + 1)\frac{d}{dt}$ (see Proposition 3.5).

For a complex number $\alpha \in \mathbb{C}$ such that $\operatorname{Re}\alpha > -1$, we write $L_\ell^{(\lambda)}$ for the Laguerre polynomial defined by

$$L_\ell^{(\lambda)}(t) := \frac{(\lambda+1)_\ell}{\ell!}\sum_{j=0}^{\ell}\frac{(-\ell)_j}{(\alpha+1)_j}\frac{t^j}{j!}$$

$$= \sum_{j=0}^{\ell}\frac{(-1)^j\Gamma(\lambda+\ell+1)}{(\ell-j)!\Gamma(\lambda+j+1)}t^j.$$

$L_\ell^{(\lambda)}(t)$ is the unique polynomial of degree ℓ satisfying the Laguerre differential equation

$$\Big(t\frac{d^2}{dt^2} + (\lambda + 1 - t)\frac{d}{dt} + \ell\Big)f(t) = 0 \qquad (3.8)$$

and

$$f^{(\ell)}(0) = (-1)^\ell.$$

Proposition 3.5. *For any $c \neq 0$ and $\ell \in \mathbb{N}$,*

$$\exp\Big(-c(t\frac{d^2}{dt^2}+(\lambda+1)\frac{d}{dt})\Big)t^\ell = (-c)^\ell \ell! L_\ell^{(\lambda)}\Big(\frac{t}{c}\Big). \tag{3.9}$$

Since the differential operator $B_t := t\frac{d^2}{dt^2} + (\lambda+1)\frac{d}{dt}$ is homogeneous of degree -1, namely, $B_t = cB_x$ if $x = ct$, it is sufficient to prove Proposition 3.5 in the case $c = 1$.

3.3. **Construction of an orthonormal basis in $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.**

We recall from (1.2) and (2.2) that the weight function $\vartheta_{k,a}$ on $\mathbb{R}^N$ is given by

$$\vartheta_{k,a}(x) = \|x\|^{a-2} \prod_{\alpha\in\mathscr{R}^+} |\langle \alpha, x\rangle|^{2k_\alpha} = \|x\|^{a-2}\vartheta_k(x),$$

and therefore

$$\begin{aligned}\vartheta_{k,a}(x)dx &= \vartheta_{k,a}(r\omega)r^{N-1}drd\sigma(\omega)\\ &= r^{2\langle k\rangle+N+a-3}\vartheta_k(\omega)drd\sigma(\omega)\end{aligned}$$

with respect to the polar coordinate $x = r\omega$ ($r > 0$, $\omega \in S^{N-1}$), where $d\sigma(\omega)$ is the standard measure on the unit sphere. Accordingly, we have a unitary isomorphism:

$$L^2(S^{N-1}, \vartheta_k(\omega)d\sigma(\omega))\widehat{\otimes}L^2(\mathbb{R}_+, r^{2\langle k\rangle+N+a-3}dr) \xrightarrow{\sim} L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx),$$

where $\widehat{\otimes}$ stands for the Hilbert completion of the tensor product space of two Hilbert spaces. Hence, we get the irreducible decomposition theorem of the $\mathfrak{sl}_2$ representation on (a dense subspace of) $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ by finding an orthogonal basis for

$$L^2(\mathbb{R}_+, r^{2\langle k\rangle+N+a-3}dr).$$

Combining, we get a direct sum decomposition of the Hilbert space:

$$\sum_{m\in\mathbb{N}}^{\oplus}(\mathscr{H}_k^m(\mathbb{R}^N)|_{S^{N-1}})\otimes L^2(\mathbb{R}_+, r^{2\langle k\rangle+N+a-3}dr) \xrightarrow{\sim} L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx).$$

For $\ell, m \in \mathbb{N}$ and $p \in \mathscr{H}_k^m(\mathbb{R}^N)$, we introduce the following functions of $x = r\omega \in \mathbb{R}^N$ ($r > 0$, $\omega \in S^{N-1}$) by

$$\Phi_\ell^{(a)}(p,x) := p(x) L_\ell^{(\lambda_{k,a,m})}\Big(\frac{2}{a}\|x\|^a\Big) \exp\Big(-\frac{1}{a}\|x\|^a\Big) \tag{3.10}$$

$$= p(\omega) r^m L_\ell^{(\lambda_{k,a,m})}\Big(\frac{2}{a} r^a\Big) \exp\Big(-\frac{1}{a} r^a\Big)$$

where we set

$$\lambda_{k,a,m} := \frac{2m + 2\langle k\rangle + N - 2}{a}. \tag{3.11}$$

Our functions $\Phi_\ell^{(a)}(p,x)$ include the well-studied functions as special cases:

$$a = 2$$
$$k \equiv 0,\ N = 1$$
$$k \equiv 0,\ a = 1.$$

We introduce the following vector space of functions on $\mathbb{R}^N$ by

$$W_{k,a}(\mathbb{R}^N) :=$$

$\mathbb{C}\text{-span}\{\Phi_\ell^{(a)}(p,\cdot) \mid \ell \in \mathbb{N}, m \in \mathbb{N}, p \in \mathscr{H}_k^m(\mathbb{R}^N)\}$.

Proposition 3.6. *Suppose k is a multiplicity function on the root system $\mathscr{R}$ and $a > 0$ such that*

$$a + 2\langle k\rangle + N - 2 > 0. \tag{3.12}$$

Let $\ell, s, m, n \in \mathbb{N}$, $p \in \mathscr{H}_k^m(\mathbb{R}^N)$ and $q \in \mathscr{H}_k^n(\mathbb{R}^N)$.

1) $\Phi_\ell^{(a)}(p,x) \in L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.

2)

$$\int_{\mathbb{R}^N} \Phi_\ell^{(a)}(p,x)\Phi_s^{(a)}(q,x)\vartheta_{k,a}(x)dx = \delta_{m,n}\delta_{\ell,s} \frac{a^{\lambda_{k,a,m}}\Gamma(\lambda_{k,a,m}+\ell+1)}{2^{1+\lambda_{k,a,m}}\Gamma(\ell+1)} \int_{S^{N-1}} p(\omega)q(\omega)\vartheta_k(\omega)d\sigma(\omega).$$

3) $W_{k,a}(\mathbb{R}^N)$ *is a dense subspace of*

$$L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$$

.

Proposition 3.7. *We fix $m \in \mathbb{N}$, $a > 0$, and a multiplicity function k satisfying*

$$2m + 2\langle k\rangle + N + a - 3 > 0.$$

For each $\ell \in \mathbb{N}$, we set

$$f^{(a)}_{\ell,m}(r) := \Big(\frac{2^{\lambda_{k,a,m}+1}\Gamma(\ell+1)}{a^{\lambda_{k,a,m}}\Gamma(\lambda_{k,a,m}+\ell+1)}\Big)^{1/2} r^m L^{(\lambda_{k,a,m})}_{\ell}\Big(\frac{2}{a}r^a\Big)\exp(-\frac{1}{a}r^a).$$

Then $\{f^{(a)}_{\ell,m}(r) : \ell \in \mathbb{N}\}$ forms an orthonormal basis in $L^2(\mathbb{R}_+, r^{2\langle k\rangle+N+a-3}dr)$.

For each $m \in \mathbb{N}$, we take an orthonormal basis $\{h^{(m)}_j\}_{j\in J_m}$ of the space $\mathscr{H}^m_k(\mathbb{R}^N)$. Proposition 3.6 immediately yields the following statement.

Corollary 3.8. *Suppose $a > 0$ and k satisfy the inequality* (3.12). *For $\ell, m \in \mathbb{N}$ and $j \in J_m$, we set*

$$\Phi^{(a)}_{\ell,m,j}(x) := h^{(m)}_j\Big(\frac{x}{\|x\|}\Big) f^{(a)}_{\ell,m}(\|x\|).$$

Then, the set $\left\{\Phi^{(a)}_{\ell,m,j} \mid \ell \in \mathbb{N}, m \in \mathbb{N}, j \in J_m\right\}$ *forms an orthonormal basis of* $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.

Now we are ready to exhibit the action of the $\mathfrak{sl}_2$ triple $\{\mathbf{k}, \mathbf{n}^+, \mathbf{n}^-\}$ on the vector space $W_{k,a}$.

Theorem 3.9. $W_{k,a}$ *is stable under the action of* $\mathfrak{sl}(2,\mathbb{C})$. *More precisely, for each fixed* $p \in \mathscr{H}^m_k(\mathbb{R}^N)$, *the action* $\omega_{k,a}$ *(see* (3.5 a–c)*) is given as follows:*

$$\omega_{k,a}(\mathbf{k})\Phi^{(a)}_\ell(p,x) = (2\ell + \lambda_{k,a,m} + 1)\Phi^{(a)}_\ell(p,x), \tag{3.13 a}$$

$$\omega_{k,a}(\mathbf{n}^+)\Phi^{(a)}_\ell(p,x) = -i(\ell+1)\Phi^{(a)}_{\ell+1}(p,x), \tag{3.13 b}$$

$$\omega_{k,a}(\mathbf{n}^-)\Phi^{(a)}_\ell(p,x) = -i(\ell + \lambda_{k,a,m})\Phi^{(a)}_{\ell-1}(p,x), \tag{3.13 c}$$

where $\Phi^{(a)}_\ell(p,x)$ *is defined in* (3.10) *and* $\lambda_{k,a,m} = (2m + 2\langle k\rangle + N - 2)/a$ *(see* (3.11)*). We have used the convention* $\Phi^{(a)}_{-1} \equiv 0$.

Theorem 3.9 may be visualized by the diagram below. We see that for each fixed k, a, and $p \in \mathscr{H}_k^m(\mathbb{R}^N)$, the operators $\omega_{k,a}(\mathbf{n}^+)$ and $\omega_{k,a}(\mathbf{n}^-)$ act as *shift operators*.

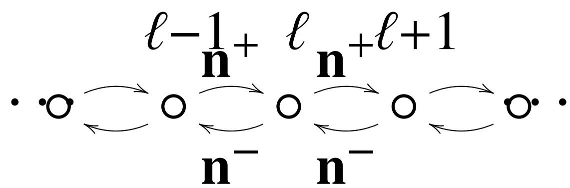

DIAGRAM 3.3.

Here, the dots represent $\omega_{k,a}(\mathbf{k})$ eigenvectors $\Phi_\ell^{(a)}(p, x)$ arranged by increasing $\omega_{k,a}(\mathbf{k})$ eigenvalues, from left to right.

We recall that an operator T is called essentially self-adjoint, if it is symmetric and its closure is self-adjoint.

Corollary 3.10. *Let $a > 0$ and k be a non-negative multiplicity function.*

(1) *The differential-difference operator $\Delta_{k,a} = \|x\|^{2-a}\Delta_k - \|x\|^a$ is an essentially self-adjoint operator on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.*

(2) *There is no continuous spectrum of* $\Delta_{k,a}$.

(3) *The set of discrete spectra of* $-\Delta_{k,a}$ *is given by*

$$\{2a\ell + 2m + 2\langle k\rangle + N - 2 + a : \ell, m \in \mathbb{N}\} \quad (N \geq 2),$$
$$\{2a\ell + 2\langle k\rangle + a \pm 1 : \ell \in \mathbb{N}\} \quad (N = 1).$$

Proof. In light of the formula (3.6 a)

$$\Delta_{k,a} = -a\omega_{k,a}(\mathbf{k}),$$

the eigenvalues of $\Delta_{k,a}$ are read from Theorem 3.9. Since $W_{k,a}(\mathbb{R}^N)$ is dense in

$$L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$$

(see Proposition 3.6), the remaining statement of Corollary 3.10 is straightforward from the following general result. □

Fact 3.11. *Let* T *be a symmetric operator on a Hilbert space* $\mathscr{H}$ *with domain* $\mathbb{D}(T)$, *and let* $\{f_n\}_n$ *be a complete orthogonal set in* $\mathscr{H}$. *If each* $f_n \in \mathbb{D}(T)$ *and there exists* $\mu_n \in \mathbb{R}$ *such that* $Tf_n = \mu_n f_n$, *for every* n, *then* T *is essentially self-adjoint.*

Applying general results to our specific setting where G is the universal covering group $\widetilde{SL(2,\mathbb{R})}$ of $SL(2,\mathbb{R})$, obtain the following two theorems:

Theorem 3.12. *Suppose $a > 0$ and k is a $\mathbb{R}$-valued multiplicity function such that $a + 2\langle k\rangle + N - 2 > 0$. Then the infinitesimal representation $\omega_{k,a}$ of $\mathfrak{sl}(2,\mathbb{R})$ lifts to a unique unitary representation $\Omega_{k,a}$ of G on the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$. In particular, we have*

$$\omega_{k,a}(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Omega_{k,a}(\mathrm{Exp}(tX)), \qquad X \in \mathfrak{g},$$

on $W_{k,a}(\mathbb{R}^N)$, the dense subspace of

$$L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx).$$

Here, we have written Exp *for the exponential map of $\mathfrak{sl}(2,\mathbb{R})$ into G.*

Theorem 3.13. *Retain the assumption of Theorem 3.12. Then, as a representation of the* *direct product group* $\mathfrak{C}\times\widetilde{SL(2,\mathbb{R})}$*, the unitary representation* $L^2(\mathbb{R}^N,\vartheta_{k,a}(x)dx)$ *decomposes discretely as*

$$L^2(\mathbb{R}^N,\vartheta_{k,a}(x)dx) = \sum_{m=0}^{\infty}{}^{\oplus}\Big(\mathscr{H}_k^m(\mathbb{R}^N)|_{S^{N-1}}\Big)\otimes\pi(\lambda_{k,a,m}).$$

Here, $\lambda_{k,a,m}=\frac{2m+2\langle k\rangle+N-2}{a}$ *(see* (3.11)*). The decomposition of the Hilbert space*

$$L^2(\mathbb{R}^N,\vartheta_{k,a}(x)dx)$$

is given by (2.8) *In particular, the summands are mutually orthogonal with respect to the inner product on* $L^2(\mathbb{R}^N,\vartheta_{k,a}(x)dx)$*.*

A map u of a real Lie group G into a Hilbert space $\mathscr{H}$ is said to be analytic at a point $g_0\in G$ if there exists a neighborhood V of g_0, an analytic coordinate system $t_1(g),\dots,t_l(g)$ on V, and coefficients $\psi_{\mathbf{n}}\in\mathscr{H}$, $\mathbf{n}\in\mathbb{N}_0^l$, such that $\sum_{\mathbf{n}\in\mathbb{N}_0^l} \|\psi_{\mathbf{n}}\||t(g)^{\mathbf{n}}| < \infty$ and

$$u(g) = \sum_{\mathbf{n}\in\mathbb{N}_0^l} \psi_{\mathbf{n}} t(g)^{\mathbf{n}}$$

for all $g \in V$. Here $l = \dim(\mathfrak{g})$ and $t(g)^{\mathbf{n}} = t_1(g)^{n_1} \cdots t_l(g)^{n_l}$. The map u is said to be analytic on G if u is analytic at each point g_0 in G. A vector v in $\mathscr{H}$ is called an analytic vector for a unitary representation Ω of G in $\mathscr{H}$ if the map $g \mapsto \Omega(g)v$ of G into $\mathscr{H}$ is analytic on G in the sense just defined.

Recall from the proof of Theorem 3.12 that the elements $\Phi_\ell^{(a)}(p)$ are eigenfunctions of $\omega_{k,a}(\Box)$, where $\Box := \mathbf{u_1}^2 + \mathbf{u_2}^2 + \mathbf{u_3}^2$. It follows that these eigenfunctions are analytic vectors for $\omega_{k,a}(\Box)$. Hence, we have:

Proposition 3.14. *$W_{k,a}$ is dense in the space of analytic vectors of the unitary representation $\Omega_{k,a}$.*

3.4. **Connection with the Gelfand–Gindikin program.**

We recall that $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ is a basis of $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$. We set

$$W := \{X = t_1\mathbf{u_1}+t_2\mathbf{u_2}+t_3\mathbf{u_3} : t_1 \geq 0,\ t_3^2+t_2^2-t_1^2 \leq 0\}.$$

Then W is an invariant closed cone in $\mathfrak{g}$ and is expressed as

$$\begin{aligned} W &= \mathrm{Ad}(SL(2,\mathbb{R}))\mathbb{R}_{\geq 0}\mathbf{u}_1 \\ &= i\,\mathrm{Ad}(SL(2,\mathbb{R}))\mathbb{R}_{\geq 0}\mathbf{k}. \end{aligned}$$

Here, $\mathbf{u}_1 = i\mathbf{k} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{g}$ and $\mathbb{R}_{\geq 0} = \{t \in \mathbb{R} : t \geq 0\}$.

We write $\exp_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} \to SL(2,\mathbb{C})$ for the exponential map. Its restriction to iW is an injective map, and we define the following subset $\Gamma(W)$ of $SL(2,\mathbb{C})$ by

$$\Gamma(W) := SL(2,\mathbb{R})\exp_{\mathbb{C}}(iW).$$

Then, $\Gamma(W)$ becomes a semigroup (the *Olshanski semigroup*).

Denote by $\widetilde{\Gamma(W)}$ the universal covering semigroup of $\Gamma(W)$, and write

$$\operatorname{Exp} : \mathfrak{g} + iW \to \widetilde{\Gamma(W)}$$

for the lifting of $\exp_{\mathbb{C}\,|\mathfrak{g}+iW} : \mathfrak{g}+iW \to \Gamma(W)$. Then $\widetilde{\Gamma(W)} = S\widetilde{L(2,\mathbb{R})}\operatorname{Exp}(iW)$ and the polar map

$$S\widetilde{L(2,\mathbb{R})} \times W \to \widetilde{\Gamma(W)},\ \ (g, X) \mapsto g\operatorname{Exp}(iX)$$

is a homeomorphism.

Since W is an $\operatorname{Ad}(SL(2,\mathbb{R}))$-invariant cone, $\Gamma(W)$ is invariant under the action of $SL(2,\mathbb{R})$ from the left and right. Thus, the semigroup $\Gamma(W)$ is written also as

$$\Gamma(W) = SL(2,\mathbb{R})\exp_{\mathbb{C}}(-\mathbb{R}_{\geq 0}\mathbf{k})SL(2,\mathbb{R}).$$

Its interior is given by

$$\Gamma(W^0) = SL(2,\mathbb{R})\exp(-\mathbb{R}_{\geq 0}\mathbf{k})SL(2,\mathbb{R}).$$

and, we have

$$\widetilde{\Gamma(W)} = S\widetilde{L(2,\mathbb{R})}\exp_{\mathbb{C}}(-\mathbb{R}_{\geq 0}\mathbf{k})S\widetilde{L(2,\mathbb{R})}.$$

By Theorem 3.9, $\Omega_{k,a}$ is a discretely decomposable unitary representation of $\widetilde{SL(2,\mathbb{R})}$ on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$. It has a lowest weight $(2\langle k\rangle + N + a - 2)/a$. It then follows that $\Omega_{k,a}$ extends to a representation of the Olshanski semigroup $\widetilde{\Gamma(W)}$, denoted by the same symbol, such that:

(P1) $\Omega_{k,a} : \widetilde{\Gamma(W)} \to \mathscr{B}(L^2)$ is strongly continuous semigroup homomorphism.
(P2) For all $f \in L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$, the map $\gamma \mapsto \langle\!\langle \Omega_{k,a}(\gamma)f, f\rangle\!\rangle_k$ is holomorphic on $\widetilde{\Gamma(W^0)}$.
(P3) $\Omega_{k,a}(\gamma)^* = \Omega_{k,a}(\gamma^\sharp)$, where $\gamma^\sharp = \mathrm{Exp}(iX)g^{-1}$ for $\gamma = g\,\mathrm{Exp}(iX)$.

Here, we have denoted by $\mathscr{B}(L^2)$ the space of bounded operators on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.

4. The integral representation of the holomorphic semigroup $\Omega_{k,a}(\gamma_z)$

The goal here is to prove an explicit integral formula of the semigroup $\Omega_{k,a}(\gamma_z)$ in the Schrödinger model $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$, for $z \in \mathbb{C}^+ \setminus i\pi\mathbb{Z}$.

4.1. **Integral representation for $\Omega_{k,a}(\gamma_z)$.**

For $\nu \in \mathbb{C}$, we set

$$\widetilde{I}_\nu(w) := \left(\frac{w}{2}\right)^{-\nu} I_\nu(w) = \sum_{\ell=0}^{\infty} \frac{w^{2\ell}}{2^{2\ell}\ell!\Gamma(\nu+\ell+1)} \tag{4.1}$$

$$= \frac{1}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_{-1}^{1} e^{wt}(1-t^2)^{\nu-\frac{1}{2}} dt. \tag{4.2}$$

The point is that we can express the relevant kernels in rank one in terms of these; and in higher rank similar functions applied to the radial variable.

5. The (k, a)-generalized Fourier transforms $\mathscr{F}_{k,a}$

The object of this section is the (k, a)-generalized Fourier transform given by

$$\mathscr{F}_{k,a} = e^{\frac{\pi i}{2}(\frac{2\langle k\rangle+N+a-2}{a})} \exp\Big(\frac{\pi i}{2a}(\|x\|^{2-a}\Delta_k - \|x\|^a)\Big).$$

This is a unitary operator on the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.

As we mentioned in Introduction, the unitary operator $\mathscr{F}_{k,a}$ includes some known transforms as special cases:

- the Euclidean Fourier $(a = 2, k \equiv 0)$,
- the Hankel transform $(a = 1, k \equiv 0)$,
- the Dunkl transform $\mathscr{D}_k$ $(a = 2, k > 0)$.

In this section, we study the unitary operators $\mathscr{F}_{k,a}$ in details for general a and k by using the idea of $\mathfrak{sl}_2$-triple. The point here is that we can interpret $\mathscr{F}_{k,a}$ not as an isolated operator but as a special value of the unitary representation $\Omega_{k,a}$ of the simply connected, simple Lie group $\widetilde{SL(2,\mathbb{R})}$ at $\operatorname{Exp} \frac{\pi}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and also as the boundary value of the holomorphic semigroup. Then, we see that some of basic properties of the Euclidean Fourier transforms can be extended to our operators $\mathscr{F}_{k,a}$ by using the representation theory of $\widetilde{SL(2,\mathbb{R})}$. Our theorem for $\mathscr{F}_{k,a}$ includes the inversion formula, and a generalization of the Plancherel formula, the Hecke formula, the Bochner formula, and the Heisenberg inequality for the uncertainty principle.

As in the Introduction, the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{k,a}($ admits much higher symmetries than $\mathfrak{C}\times\widetilde{SL(2,\mathbb{R})}$ for particular values of (k, a). In fact, if $k \equiv 0$, then the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ is a representation space of the Schrödinger model of the Weil representation of the metaplectic group $Mp(N, \mathbb{R})$ for $a = 2$, and the L^2-model of the minimal representation of the conformal group $O(N+1,2)$ for $a = 1$. The special value $a = 2$ has been paid much attention also for $k > 0$ in the sense that $\mathscr{F}_{k,2}$ is nothing but the Dunkl operator $\mathscr{D}_k$. How about the $a = 1$ case for general $k > 0$? In this section, we analyze the unitary operator

$$\mathscr{H}_k := \mathscr{F}_{k,1} \tag{5.1}$$

in a more concrete form. The unitary operator $\mathscr{H}_k$ may be regarded as the Dunkl analogue of the classical Hankel transform $\mathscr{F}_{0,1}$. We shall give an explicit kernel of $\mathscr{H}_k$ by means of the Dunkl transform V_k and the classical Bessel functions.

5.1. $\mathscr{F}_{k,a}$ as an inversion unitary element.

The object of our study in this section is a (k, a)-generalized Fourier transform $\mathscr{F}_{k,a}$ defined as

$$\mathscr{F}_{k,a}(f) := e^{i\frac{\pi}{2}(\frac{2\langle k\rangle+N+a-2}{a})}\Omega_{k,a}(\gamma_{i\frac{\pi}{2}})f, \tag{5.2}$$

for $f \in L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$. Here, we recall from that

$$\gamma_{\frac{\pi i}{2}} = \operatorname{Exp}\Big(\frac{\pi}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\Big)$$

is defined as an element of the simply connected Lie group $\widetilde{SL(2,\mathbb{R})}$, and from Theorem 3.12 that $\Omega_{k,a}$ is a unitary representation of $\widetilde{SL(2,\mathbb{R})}$ on the Hilbert space $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$.

In this subsection, we discuss basic properties of $\mathscr{F}_{k,a}$ for general k and a, which are derived from the simple fact that $\gamma_{\frac{\pi i}{2}}$ gives the non-trivial (therefore, the longest) element of Weyl group with respect to the $\mathfrak{sl}_2$-triple $\{\mathbf{h}, \mathbf{e}^+, \mathbf{e}^-\}$.

Theorem 5.1. *Let $a > 0$ and k be a non-negative multiplicity function on the root system $\mathscr{R}$.*

(1) (Plancherel formula) *The (k,a)-generalized Fourier transform $\mathscr{F}_{k,a} : L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx) \to L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ is a unitary operator.*

That is, $\mathscr{F}_{k,a}$ is a bijective linear operator satisfying

$$\|\mathscr{F}_{k,a}(f)\|_k = \|f\|_k \quad \text{for any } f \in L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx).$$

(2) *We recall from* (3.10) *that $\Phi_\ell^{(a)}(p,\cdot)$ is defined as*

$$\Phi_\ell^{(a)}(p,x) = p(x)L_\ell^{(\lambda_{k,a,m})}\Big(\frac{2}{a}\|x\|^a\Big)\exp\Big(-\frac{1}{a}\|x\|^a\Big),$$

for $\ell, m \in \mathbb{N}$ and $p \in \mathscr{H}_k^m(\mathbb{R}^N)$. Then, $\Phi_\ell^{(a)}(p,\cdot)$ is an eigenfunction of the (k,a)-generalized Fourier transform $\mathscr{F}_{k,a}$:

$$\mathscr{F}_{k,a}(\Phi_\ell^{(a)}(p,\cdot)) = e^{-i\pi(\ell+\frac{m}{a})}\Phi_\ell^{(a)}(p,\cdot). \tag{5.3}$$

Proof. The first statement is an immediate consequence of the fact that $\Omega_{k,a}$ is a unitary representation of $\widetilde{SL(2,\mathbb{R})}$.

To see the second statement, we recall from Theorem 3.9 that $\Phi_\ell^{(a)}(p,\cdot)$ is an eigenfunction of $\omega_{k,a}(\mathbf{k})$. Then, the integration of (3.13 a) shows the identity (5.3). □

Corollary 5.2. *The (k,a)-generalized Fourier transform $\mathscr{F}_{k,a}$ is of finite order if and only if $a \in \mathbb{Q}$. If a is of the form $a = \frac{q}{q'}$, where q and q' are positive integers such that $(q,q') = 1$, then*

$$(\mathscr{F}_{k,a})^{2q} = \mathrm{id}\,.$$

In particular $(\mathscr{F}_{k,1})^2 = \mathrm{id}$ *and* $(\mathscr{F}_{k,2})^4 = \mathrm{id}$.

Proof. Since $\{\Phi_\ell^{(a)}(p,\cdot) \mid \ell \in \mathbb{N}, m \in \mathbb{N}, p \in \mathscr{H}_k^m(\mathbb{R}^N)\}$ spans a dense subspace in

$$L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx),$$

it follows from (5.3) that the unitary operator $\mathscr{F}_{k,a}$ is of finite order if and only if $a \in \mathbb{Q}$. If $a = \frac{q}{q'}$, then $(\mathscr{F}_{k,a})^{2q}$ acts on $\Phi_\ell^{(0)}(p,\cdot)$ as a scalar

$$\left(e^{-i\pi(\ell+\frac{m}{a})}\right)^{2q} = 1$$

for any $m \in \mathbb{N}$ and any $p \in \mathscr{H}_k^m(\mathbb{R}^N)$. Thus, we have proved $(\mathscr{F}_{k,a})^{2q} = \mathrm{id}$. $\square$

Remark 5.3. *We recall Theorem 3.13 asserting that* $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ *decomposes into a discrete direct sum of irreducible unitary representations of* $G = \widetilde{SL(2,\mathbb{R})}$*. Hence,* $\mathscr{F}_{k,a}^2$ *acts as a scalar multiple on each summand of* (**??**) *because* $\gamma_{\pi i}$ *is a central element of G and* $\mathscr{F}_{k,a}^2 = e^{i\pi(\frac{2\langle k\rangle+N+a-2}{a})}\Omega_{k,a}(\gamma_{\pi i})$ *by* (5.2)*. Since* $\gamma_{\pi i}$ *acts on the irreducible representation* $\pi(\lambda_{k,a,m})$ *as a scalar* $e^{-\pi i(\lambda_{k,a,m}+1)}$, $\mathscr{F}_{k,a}^2$ *acts on it as the scalar*

$$e^{i\pi(\frac{2\langle k\rangle+N+a-2}{a})}e^{-\pi i(\lambda_{k,a,m}+1)} = e^{-\frac{2m\pi i}{a}}.$$

This gives us an alternative interpretation of Corollary 5.2.

Next, we discuss intertwining properties of the (k, a)-generalized Fourier transform $\mathscr{F}_{k,a}$ with differential operators. Let $E = \sum_{j=1}^N x_j\partial_j$ be the Euler operator on $\mathbb{R}^N$ as before.

Theorem 5.4. *The unitary operator $\mathscr{F}_{k,a}$ satisfies the following intertwining relations on a dense subspace of $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$:*

(1) $\mathscr{F}_{k,a} \circ E = -(E + N + 2\langle k \rangle + a - 2) \circ \mathscr{F}_{k,a}$.

(2) $\mathscr{F}_{k,a} \circ \|x\|^a = -\|x\|^{2-a}\Delta_k \circ \mathscr{F}_{k,a}$.

(3) $\mathscr{F}_{k,a} \circ \|x\|^{2-a}\Delta_k = -\|x\|^a \circ \mathscr{F}_{k,a}$.

If we use ξ (instead of x) for the variables of the target space of $\mathscr{F}_{k,a}$, we may write Theorem 5.4 (2) and (3) as

$$\mathscr{F}_{k,a}(\| \cdot \|^a f)(\xi) = -\|\xi\|^{2-a}\Delta_k \mathscr{F}_{k,a}(f)(\xi), \tag{5.4 a}$$

$$\mathscr{F}_{k,a}(\| \cdot \|^{2-a}\Delta_k f)(\xi) = -\|\xi\|^a \mathscr{F}_{k,a}(f)(\xi). \tag{5.4 b}$$

Proof of Theorem 5.4. We observe that $\gamma_{\frac{\pi i}{2}}$ is a representative of the longest Weyl group element, and satisfies

$$\mathrm{Ad}(\gamma_{\frac{\pi i}{2}})\mathbf{h} = -\mathbf{h}, \ \mathrm{Ad}(\gamma_{\frac{\pi i}{2}})\mathbf{e}^+ = -\mathbf{e}^-,$$

$$\mathrm{Ad}(\gamma_{\frac{\pi i}{2}})\mathbf{e}^- = -\mathbf{e}^+,$$

(see (3.1) for the definition of $\mathbf{e}^+$, $\mathbf{e}^-$, and $\mathbf{h}$). In turn, we apply the identity

$$\Omega_{k,a}(g)\omega_{k,a}(X)\Omega_{k,a}(g)^{-1} = \omega_{k,a}(\mathrm{Ad}(g)X), \quad (g \in G,\ X \in \mathfrak{g}),$$

to $\mathbb{E}^+_{k,a} = \omega_{k,a}(\mathbf{e}^+)$, $\mathbb{E}^-_{k,a} = \omega_{k,a}(\mathbf{e}^-)$, $\mathbb{H}_{k,a} = \omega_{k,a}(\mathbf{h})$ (see (**??**)) and $\mathscr{F}_{k,a} = \Omega_{k,a}(\gamma_{\frac{\pi i}{2}})$. Then we have

$$\begin{aligned}
\mathscr{F}_{k,a} \circ \mathbb{H}_{k,a} &= -\mathbb{H}_{k,a} \circ \mathscr{F}_{k,a}, \qquad (5.5)\\
\mathscr{F}_{k,a} \circ \mathbb{E}^+_{k,a} &= -\mathbb{E}^-_{k,a} \circ \mathscr{F}_{k,a},\\
\mathscr{F}_{k,a} \circ \mathbb{E}^-_{k,a} &= -\mathbb{E}^+_{k,a} \circ \mathscr{F}_{k,a}.
\end{aligned}$$

Now, Theorem 5.4 follows. □

5.2. Density of (k,a)-generalized Fourier transform $\mathscr{F}_{k,a}$.

By the Schwartz kernel theorem, the unitary operator $\mathscr{F}_{k,a}$ can be expressed by means of a distribution kernel. We adopt an expression in terms of a generalized function in the sense of Gel'fand. This means that we can write the unitary operator $\mathscr{F}_{k,a}$ on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$

as an ‘integral transform’:

$$\mathscr{F}_{k,a}f(\xi) = c_{k,a} \int_{\mathbb{R}^N} B_{k,a}(x,\xi)f(x)\vartheta_{k,a}(x)dx, \tag{5.6}$$

Here, we have normalized the integral by the constant $c_{k,a}$. In light of the unitary isomorphism

$$L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx) \to L^2(\mathbb{R}^N, dx),$$

$$f(x) \mapsto f(x)\vartheta_{k,a}(x)^{\frac{1}{2}},$$

$B_{k,a}(x,\xi)\vartheta_{k,a}(x)^{\frac{1}{2}}\vartheta_{k,a}(\xi)^{\frac{1}{2}}$ is a tempered distribution on $\mathbb{R}^N \times \mathbb{R}^N$

Then, Theorem 5.4 is reformulated as the differential equations that are satisfied by the distribution kernel $B_{k,a}(x,\xi)$ as follows:

Theorem 5.5. *The distribution $B_{k,a}(\cdot,\cdot)$ solves the following differential-difference equation* *on* $\mathbb{R}^N$

$$E^x B_{k,a}(x,\xi) = E^\xi B_{k,a}(x,\xi), \qquad (5.7\text{ a})$$

$$\|\xi\|^{2-a}\Delta_k^\xi B_{k,a}(x,\xi) = -\|x\|^a B_{k,a}(x,\xi), \qquad (5.7\text{ b})$$

$$\|x\|^{2-a}\Delta_k^x B_{k,a}(x,\xi) = -\|\xi\|^a B_{k,a}(x,\xi). \qquad (5.7\text{ c})$$

Here, the superscript in E^x, Δ_k^x, *etc indicates the relevant variable.*

(The rank-one case) Recall that in the rank one case we have the expression of $\Lambda_{k,a}$ for all $a > 0$. That is, in this case we still have an explicit formula without the assumption $a = 1, 2$. From Fact **??**, the kernel $B_{k,a}$, for arbitrary $a > 0$ and $k > 0$, is given explicitly as

$$B_{k,a}(x,\lambda) = e^{i\frac{\pi}{2}(\frac{2k+a-1}{a})}\Lambda_{k,a}(x,\lambda; i\frac{\pi}{2})$$

$$= \Gamma\Big(\frac{2k+a-1}{a}\Big)\Big(\widetilde{J}_{(2k-1)/a}\Big(\frac{2}{a}|x\lambda|^{\frac{a}{2}}\Big)+$$

$$\frac{x\lambda}{(ai)^{\frac{2}{a}}}\widetilde{J}_{(2k+1)/a}\Big(\frac{2}{a}|x\lambda|^{\frac{a}{2}}\Big)\Big),$$

where

$$\widetilde{J}_\nu(w)=\Big(\frac{w}{2}\Big)^{-\nu}J_\nu(w).$$

5.3. **Master formula and its applications.**

Let $a>0$ and k be a non-negative multiplicity function on the root system $\mathscr{R}$. We recall that

$$\mathbb{E}^+_{k,a}=\omega_{k,a}\begin{pmatrix}0&1\\0&0\end{pmatrix}=\frac{i}{a}\|x\|^a,$$

$$\mathbb{E}^-_{k,a}=\omega_{k,a}\begin{pmatrix}0&0\\1&0\end{pmatrix}=\frac{i}{a}\|x\|^{2-a}\Delta_k$$

are infinitesimal generators of the unitary representation $\Omega_{k,a}$ of $G=S\widetilde{L(2,\mathbb{R})}$ on

$$L^2(\mathbb{R}^N,\vartheta_{k,a}(x)dx)$$

.

We introduce the operator

$$\mathscr{B}_{k,a}:=\exp(i\,\mathbb{E}^+_{k,a})\exp\Big(\frac{i}{2}\mathbb{E}^-_{k,a}\Big).\tag{5.8}$$

We set

$$c_0 := \operatorname{Exp} i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \operatorname{Exp} \frac{i}{2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in G.$$

$$\operatorname{Ad}(c_0)\, \mathbf{h} = -\mathbf{k} \tag{5.9}$$

because

$$\operatorname{Ad}(c_0)\, \mathbf{h} = \begin{pmatrix} \frac{1}{2} & i \\ \frac{i}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & i \\ \frac{i}{2} & 1 \end{pmatrix}^{-1} = -i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The identity (5.9) in $\mathfrak{sl}_2$ leads us to the identity

$$\mathscr{B}_{k,a} \circ \omega_{k,a}(\mathbf{h}) = -\omega_{k,a}(\mathbf{k}) \circ \mathscr{B}_{k,a}.$$

Since $\mathbb{H}_{k,a} = \omega_{k,a}(\mathbf{h})$ acts on homogeneous functions as scalar, the application of $\mathscr{B}_{k,a}$ to homogeneous functions should yield eigenfunctions of $\omega_{k,a}(\mathbf{k})$. Here is an explicit formula

Proposition 5.6. *For $\ell, m \in \mathbb{N}$ and $p \in \mathscr{H}_k^m(\mathbb{R}^N)$,*

$$\mathscr{B}_{k,a}(p(x)\|x\|^{a\ell}) = \left(-\frac{2}{a}\right)^{\ell} \ell!\, \Phi_\ell^{(a)}(p, x).$$

$$\mathscr{P}_a(\mathbb{R}^N) := \mathbb{C}\text{-span}\{p(x)\|x\|^{a\ell} : p \in \mathscr{H}_k^m(\mathbb{R}^N)\}. \tag{5.10}$$

We note that $\mathscr{P}_2(\mathbb{R}^N)$ is the space of polynomials on $\mathbb{R}^N$.

Denote by $(e^{-i\frac{\pi}{a}})^*$ the map defined as

$$(e^{-i\frac{\pi}{a}})^* p(x) := p(e^{-i\frac{\pi}{a}} x).$$

The following lemma is needed for later use.

Proposition 5.7. *Then the following diagram commutes*

$$\begin{array}{ccc} \mathscr{P}_a(\mathbb{R}^N) & \xrightarrow{\mathscr{B}_{k,a}} & L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx) \\ {\scriptstyle (e^{-i\frac{\pi}{a}})^*}\downarrow & & \downarrow{\scriptstyle \mathscr{F}_{k,a}} \\ \mathscr{P}_a(\mathbb{R}^N) & \xrightarrow{\mathscr{B}_{k,a}} & L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx) \end{array}$$

Proof. The identity (5.9) in $\mathfrak{sl}_2$ lifts to the identity

$$\mathscr{B}_{k,a} \circ \Omega_{k,a}(\operatorname{Exp} -t\,\mathbf{h}) = \Omega_{k,a}(\operatorname{Exp} t\,\mathbf{k}) \circ \mathscr{B}_{k,a},$$

and in particular

$$\mathscr{B}_{k,a}\circ\Omega_{k,a}(\operatorname{Exp} -\frac{\pi}{2i}\mathbf{h}) = \Omega_{k,a}(\operatorname{Exp} \frac{\pi}{2i}\mathbf{k})\circ\mathscr{B}_{k,a},$$

when the both-hand sides make sense on $\mathscr{P}_a(\mathbb{R}^N)$. In terms of the (k, a)-generalized Fourier transform $\mathscr{F}_{k,a}$ (see (5.2)), we get

$$\mathscr{B}_{k,a}\circ\exp\Big(\frac{\pi i}{2}\frac{2\langle k\rangle + N + a - 2}{a}\Big)\,\Omega_{k,a}\Big(\operatorname{Exp} -\frac{\pi}{2i}\mathbf{h}\Big)$$

$$= \mathscr{F}_{k,a}\circ\mathscr{B}_{k,a}.$$

On the other hand, we recall that

$$\omega_{k,a}(\mathbf{h}) = \frac{2}{a}\sum_{j=1}^{N} x_j\partial_j + \frac{N + 2\langle k\rangle + a - 2}{a},$$

and therefore its lift to the group representation is given by

$$(\Omega_{k,a}(\operatorname{Exp} t\,\mathbf{h})f)(x) = \exp\Big(\frac{N + 2\langle k\rangle + a - 2}{a}t\Big)$$

$$f(e^{\frac{2t}{a}}x).$$

Substituting $t = \frac{\pi i}{2}$, we ge $\mathscr{B}_{k,a} \circ \left(e^{-i\frac{\pi}{a}}\right)^* = \mathscr{F}_{k,a} \circ \mathscr{B}_{k,a}$. This completes the proof of Claim 5.7 □

Corollary 5.8. (Hecke type identity) *If in addition p is k-harmonic of degree m, then*

$$\mathscr{F}_{k,a}(e^{-\frac{\|\cdot\|^a}{a}} p)(\lambda) = e^{-i\frac{\pi}{a}m} e^{-\frac{1}{a}\|\lambda\|^a} p(\lambda). \tag{5.11}$$

Corollary 5.8 may be regarded as a Hecke type identity for the (k, a)-generalized Fourier transform $\mathscr{F}_{k,a}$. Another way to prove this identity is to substitute 0 for ℓ in (5.3).

The above identity is a particular case of Theorem 5.9. For this, we denote by $\boldsymbol{H}_\nu$ the classical Hankel transform of one variable defined by

$$\boldsymbol{H}_\nu(\psi)(s) := \int_0^\infty \psi(r) \widetilde{J}_\nu\Big(\frac{2}{a}(rs)^{\frac{a}{2}}\Big) r^{\frac{a}{2}\nu} dr, \tag{5.12}$$

for a function ψ defined on $\mathbb{R}_+$. Here, $\widetilde{J}_\nu$ is the normalized Bessel function $\widetilde{J}_\nu(w) = (\frac{w}{2})^{-\nu} J_\nu(w)$. Then the (k,a)-generalized Fourier transform $\mathscr{F}_{k,a}$ satisfies the following identity:

Theorem 5.9. (Bochner type identity) *If $f \in (L^1 \cap L^2)(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$ is of the form $f(x) = p(x)\psi(\|x\|)$ for some $p \in \mathscr{H}_k^m(\mathbb{R}^N)$ and a one-variable function $\psi(r)$ on $\mathbb{R}_+$, then*

$$\mathscr{F}_{k,a}(f)(\lambda) = a^{-(\frac{2m+2\langle k\rangle+N-2}{a})} e^{-i\frac{\pi}{a}m} p(\lambda) \boldsymbol{H}_{\frac{2m+2\langle k\rangle+N-2}{a}}(\psi)(\|\lambda\|),$$

In particular, if f is radial, then $\mathscr{F}_{k,a}(f)$ is also radial.

5.4. **The uncertainty principle for the transform $\mathscr{F}_{k,a}$.**

The Heisenberg uncertainty principle may be formulated by means of the so-called Heisenberg inequality for the Euclidean Fourier transform on $\mathbb{R}$. Loosely, the more a function is concentrated, the more its Fourier transform is spread. In this section we extend the Heisenberg inequality to a (k, a)-generalized Fourier transform $\mathscr{F}_{k,a}$ on $\mathbb{R}^N$.

Let k be a non-negative root multiplicity function and $k > 0$. We recall that $\|\cdot\|_k$ denotes the L^2-norm with respect to the measure $\vartheta_{k,a}(x)dx$ on $\mathbb{R}^N$ (see (1.2)). Then the goal of this subsection is to prove the following multiplicative inequality:

Theorem 5.10. (Heisenberg type inequality) *The (k, a)-generalized Fourier transform $\mathscr{F}_{k,a}$ satisfies*

$$\Big\| \|\cdot\|^{\frac{a}{2}} f \Big\|_k \, \Big\| \|\cdot\|^{\frac{a}{2}} \mathscr{F}_{k,a}(f) \Big\|_k \geq \Big(\frac{2\langle k\rangle + N + a - 2}{2} \Big) \|f\|_k^2, \tag{5.13}$$

for any $f \in L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$. The equality holds if and only if f is of the form $f(x) = \lambda \exp(-c\|x\|^a)$ for some $\lambda \in \mathbb{C}$ and $c \in \mathbb{R}_+$.

Remark 5.11. *The inequality* (5.13) *for* $k = 0$ *and* $a = 2$ *is the original Heisenberg inequality for the Euclidean Fourier transform. The inequality for* $k > 0$ *and* $a = 2$ *is the Heisenberg type inequality for the Dunkl transform* $\mathscr{D}_k$*, which was proved by Rösler and by Shimeno.*

In order to prove Theorem 5.10 we begin with the following additive inequality:

Lemma 5.12. 1) *For all* $f \in L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$

$$\left\| \|\cdot\|^{\frac{a}{2}} f \right\|_k^2 + \left\| \|\cdot\|^{\frac{a}{2}} \mathscr{F}_{k,a}(f) \right\|_k^2 \geq (2\langle k\rangle + N + a - 2)\|f\|_k^2. \tag{5.14}$$

2) *The equality holds in* (5.14) *if and only if* $f(x)$ *is a scalar multiple of* $\exp(-\frac{1}{a}\|x\|^a)$.

Proof. By Theorem 5.4 (3) and Theorem 5.1 (1), we get

$$\begin{aligned}\left\| \|\cdot\|^{\frac{a}{2}} \mathscr{F}_{k,a} f \right\|_k^2 &= \langle\!\langle \|x\|^a \mathscr{F}_{k,a} f, \mathscr{F}_{k,a} f \rangle\!\rangle_k \\ &= -\langle\!\langle \mathscr{F}_{k,a}(\|x\|^{2-a} \Delta_k f), \mathscr{F}_{k,a} f \rangle\!\rangle_k \\ &= -\langle\!\langle \|x\|^{2-a} \Delta_k f, f \rangle\!\rangle_k.\end{aligned}$$

Hence, the left-hand side of (5.14) equals

$$\langle\!\langle (\|x\|^a - \|x\|^{2-a} \Delta_k) f, f \rangle\!\rangle_k = \langle\!\langle -\Delta_{k,a} f, f \rangle\!\rangle_k. \tag{5.15}$$

It then follows from Corollary 3.10 that the self-adjoint operator $-\Delta_{k,a}$ has only discrete spectra, of which the minimum is $2\langle k \rangle + N - 2 + a$. Therefore, we have proved

$$(5.15) \geq (2\langle k \rangle + N - 2 + a) \|f\|_k^2.$$

Thus, the inequality (5.14) has been proved. Further, the equality holds if and only if f is an eigenfunction of $-\Delta_{k,a}$ corresponding to the minimum eigenvalue $2\langle k \rangle + N - 2 + a$, namely, f is a scalar multiple of $\exp(-\frac{1}{a}\|x\|^a)$ (i.e. by putting $\ell = m = 0$ in the formula

(3.10) of $\Phi_\ell^{(a)}(p,x)$). Hence, Lemma 5.12 has been proved. □

Proof of Theorem 5.10. Now, for $c > 0$, we set $f_c(x) := f(cx)$. Using the fact that the density $\vartheta_{k,a}$ is homogeneous of degree $2\langle k\rangle + a - 2$, we get

$$\Big\| \|\cdot\|^{\frac{a}{2}} f_c \Big\|_k^2 = c^{-2\langle k\rangle - N - 2a + 2} \Big\| \|\cdot\|^{\frac{a}{2}} f \Big\|_k^2, \tag{5.16 a}$$

and

$$\|f_c\|_k^2 = c^{-2\langle k\rangle - N - a + 2} \|f\|_k^2. \tag{5.16 b}$$

Furthermore, we lift the formula in Theorem 5.4 (1) to the formula

$$(\mathscr{F}_{k,a} f_c)(x) = c^{-(N+2\langle k\rangle + a - 2)} (\mathscr{F}_{k,a} f)\Big(\frac{x}{c}\Big),$$

from which we get

$$\Big\| \|\cdot\|^{\frac{a}{2}} \mathscr{F}_{k,a}(f_c) \Big\|_k^2 = c^{-2\langle k\rangle - N + 2} \Big\| \|\cdot\|^{\frac{a}{2}} \mathscr{F}_{k,a}(f) \Big\|_k^2. \tag{5.16 c}$$

Thus, if we substitute f_c for f in Lemma 5.12, we obtain

$$c^{-a}\Big\| \|\cdot\|^{\frac{a}{2}} f \Big\|_k^2 + c^{a}\Big\| \|\cdot\|^{\frac{a}{2}} \mathscr{F}_{k,a}(f)\Big\|_k^2 \geq (2\langle k\rangle + N + a - 2)\|$$

Obviously the minimum value of the left-hand side (as a function of $c \in \mathbb{R}_+$) is

$$2\Big\| \|\cdot\|^{\frac{a}{2}} f \Big\|_k \Big\| \|\cdot\|^{\frac{a}{2}} \mathscr{F}_{k,a}(f)\Big\|_k.$$

Hence, Theorem 5.10 has been proved. □