Howe duality for the Dunkl version of the Dirac operator. (Joint work with V. Souček and P. Somberg.)

Let us consider a finite Coxeter group G with the root system R on a Euclidean space $\mathbb{R}^n$. The Dunkl operators T_i are as we have seen quite remarkable analogues of ordinary partial derivatives $\partial_i = \frac{\partial}{\partial x_i}$ on $\mathbb{R}^n$, they form a commuting family of differential-difference operators invariant with respect to the action of the group G. Again, we shall consider only non-negative weights k. The (rational) Dunkl operators T_i are as before defined by

$$T_i(p) := \partial_i(p) + \sum_{\alpha \in R^+} k_\alpha \frac{p(x) - p(r_\alpha x)}{< \alpha, x >} \alpha_i,$$

where R^+ is a set of positive roots of R and r_α denotes the reflection corresponding to a root $\alpha \in R^+$.

The original motivation behind the definition of the Dunkl operators was a study of orthogonality structures for polynomials in several variables. But there is a lot of interesting connections of these operators to various other fields in mathematics - e.g., to integrable systems of Calogero-Moser type , to calculus of Schubert cells in Grassmannians, and to Hecke algebras. A key role in various applications is played by the Dunkl Laplacian Δ_k.

The spaces of homogeneous polynomials in its kernel form a *deformation* of the classical theory of spherical harmonics (and they reduce back to it for the trivial weight $k \equiv 0$. Their basic properties are best understood using the method of the Howe dual pair. As we have seen, the Dunkl Laplacian $E = -\frac{1}{2}\Delta$ and $F = \frac{1}{2}|x|^2$ (as the multiplication operator) generate, together with their commutator $H = [E, F]$ a Lie algebra $\mathrm{sl}(2, \mathbb{R})$. This is a deformation of the action of the Howe dual pair $(\mathrm{SO}(n), \mathrm{sl}(2, \mathbb{R}))$ used in the classical theory of spherical harmonics.

Here, we are going to discuss a deformed analogue of the Fischer decomposition for spinor-valued polynomials. The main method used in the paper is a deformation of the Howe dual pair $(\mathrm{Spin}(n), \mathrm{osp}(1|2))$ used in the classical case. We review first shortly (in Sect. 2) the classical Fischer decomposition both in scalar and spinor case. To set notation, we recall the Fischer decomposition for the Dunkl Laplacian from. In Sect. 4, we construct the deformation of the Howe dual pair and of the Fischer decomposition in the spinor case.

1.

1.1. The scalar case

The easiest way how to describe the Fischer decomposition for polynomials on a Euclidean space is to use the corresponding Howe dual pair. Let $\mathcal{P}$ be the space of polynomials on $\mathbb{R}^n$ with the usual action of the group $SO(n)$. Consider the Weyl algebra $\mathcal{W}_n$ of differential operators on $\mathbb{R}^n$ with polynomial coefficients acting on $\mathcal{P}$. A polynomial $p \in \mathcal{P}$ will be considered as a zero-order operator on $\mathcal{P}$.

Let $\mathbb{E} = \sum_i x_i \partial_i$ denote the Euler operator on $\mathcal{P}$. The Laplace operator $E = -\frac{1}{2}\Delta$ and the multiplication operator $F = \frac{1}{2}|x|^2$, together with the operator $H = [E, F] = \mathbb{E} + \frac{n}{2}$, generate a copy of $\mathrm{sl}(2, \mathbb{R})$ in $\mathcal{W}_n$.

The Fischer decomposition of $\mathcal{P}$ is described by the infinite triangle below.

$$
\begin{array}{ccccccc}
\mathcal{P}_0 & \mathcal{P}_1 & \mathcal{P}_2 & \mathcal{P}_3 & \mathcal{P}_4 & \mathcal{P}_5 & \mathcal{P}_6 \\
\\
\mathcal{H}_0 = \langle 1 \rangle & \stackrel{\Delta}{\leftarrow} & \langle r^2 \rangle & \stackrel{\Delta}{\leftarrow} & \langle r^4 \rangle & \stackrel{\Delta}{\leftarrow} & \langle r^6 \rangle \\
\\
 & \mathcal{H}_1 & \stackrel{\Delta}{\leftarrow} & r^2\mathcal{H}_1 & \stackrel{\Delta}{\leftarrow} & r^4\mathcal{H}_1 & \\
\\
 & & \mathcal{H}_2 & \stackrel{\Delta}{\leftarrow} & r^2\mathcal{H}_2 & \stackrel{\Delta}{\leftarrow} & r^4\mathcal{H}_2 \\
\\
 & & & \mathcal{H}_3 & \stackrel{\Delta}{\leftarrow} & r^2\mathcal{H}_3 & \\
\\
 & & & & \mathcal{H}_4 & \stackrel{\Delta}{\leftarrow} & r^2\mathcal{H}_4
\end{array}
$$

where $r^2 = |x|^2$. Here $\mathcal{P}_n$ denote the spaces of homogeneous polynomials of degree n and each column describes the decomposition of $\mathcal{P}_n$ into a direct sum of corresponding subspaces. The left-down diagonal gives the decomposition of the space of harmonic functions $\mathcal{H}$ on $\mathbb{R}^n$ into the sum of spherical harmonic: $\mathcal{H} = \oplus_n \mathcal{H}_n$.

Individual rows $\oplus_n |x|^{2n}\mathcal{H}_k$ are (infinite-dimensional) irreducible $\mathrm{sl}(2,\mathbb{R})$-modules, the sum is a decomposition into an infinite number of isomorphic copies of an irreducible $\mathrm{SO}(n)$-module $\mathcal{H}_n$. The Laplace operator E, resp. the operator $|x|^2$, act as isomorpisms between individual summands, they shift them to the left (resp. to the right), i.e. they act as the lowering, resp. rising, operators. The decomposition of $\mathcal{P}$ into individual rows is the multiplicity free decomposition under the action of the dual pair $\mathrm{SO}(n) \times \mathrm{sl}(2,\mathbb{R})$. Any homogeneous polynomial p on $\mathbb{R}^n$ of degree d can then be clearly written (in a unique way) as a sum

$$p = \sum_{j,0 \leq j \leq fracd2} |x|^{2j} h_{d-2j};\ h_\ell \in \mathcal{H}_\ell.$$

1.2. The spinor-valued case

Many key facts from Clifford analysis are consequences of a similar Fischer decomposition for spinor-valued polynomials. Let $\mathcal{C}_{0,n}$ be the Clifford algebra of the Euclidean space $\mathbb{R}^n$ (with negative definitive quadratic form). Recall the Clifford algebra is generated by the relations

$$x^2 = -|x|^2$$

. It contains unit $1 \in \mathbb{R}$ and is generated by an orthonormal basis e_j of $\mathbb{R}^n$. Denote by S a complex irreducible module S over $\mathcal{C}_{0,n}$, the action will be denoted by the left multiplication.

Let $\mathcal{P}(S)$ denote the space of polynomials on $\mathbb{R}^n$ with values in S, and let $\mathcal{P}(S)_j$ be the subspace of homogeneous polynomials of degree j. The groups $\mathrm{Pin}(n)$ and $Spin(n)$ act on $\mathcal{P}(S)$ in a standard way as

$$[g \cdot p](x) := gp(g^{-1}xg);\ g \in \mathrm{Spin}(n) \subset \mathcal{C}_n,$$
$$x \in \mathbb{R}^n,\ p \in \mathcal{P}(S),$$

where the juxtaposition denote the Clifford multiplication.

Operators from the twisted Weyl algebra $\mathcal{W} \otimes \mathcal{C}_{0,n}$ act on $\mathcal{P}(S)$ in a natural way. The Dirac operator $\underline{D} = \sum_j x_j \partial_j$ and the operator $\underline{x} = \sum_j e_j x_j$ (acting by mutliplication and by the Clifford action on S) generate a super-Lie algebra osp$(1|2)$ in $\mathcal{W} \otimes \mathcal{C}_{0,n}$ - we shall make this explicit below. They form a basis of the odd part in osp$(1|2)$, the even part being generated by $\Delta, |x|^2$, and $\mathbb{E}$. The space $\mathcal{P}(S)$ of all spinor-valued polynomials decomposes in a multiplicity free way under the action of the Howe dual pair $\mathrm{Spin}(n) \times \mathrm{osp}(1|2)$, and the decomposition has again a form of an infinite triangle:

$$
\begin{array}{ccccccccc}
\mathcal{P}_0(S) & & \mathcal{P}_1(S) & & \mathcal{P}_2(S) & & \mathcal{P}_3(S) & & \\
 & & & & & & & & \\
\mathcal{M}_0 = \langle 1 \rangle & \stackrel{D}{\leftarrow} & \underline{x}\mathcal{M}_0 & \stackrel{D}{\leftarrow} & |x|^2\mathcal{M}_0 & \stackrel{D}{\leftarrow} & \underline{x}^3\mathcal{M}_0 & & \dots \\
 & & \oplus & & \oplus & & \oplus & & \\
 & & \mathcal{M}_1 & \stackrel{D}{\leftarrow} & \underline{x}\mathcal{M}_1 & \stackrel{D}{\leftarrow} & |x|^2\mathcal{M}_1 & & \dots \\
 & & & & \oplus & & \oplus & & \\
 & & & & \mathcal{M}_2 & \stackrel{D}{\leftarrow} & \underline{x}\mathcal{M}_2 & & \dots \\
 & & & & & & \oplus & & \\
 & & & & & & \mathcal{M}_3 & & \dots
\end{array}
$$

All summands in the same row are isomorphic as $\mathrm{Pin}(n)$-modules, and each row is an irreducible module for the Howe dual pair $\mathrm{Pin}(n) \times \mathrm{osp}(1|2)$. The Dirac operator shifts all spaces in the same row to the left, the mutliplication by $\underline{x}$ shifts them to both of these actions are isomorphisms of $\mathrm{Pin}(n)$-modules. Similarly as for harmonic polynomials, any spinor-valued polynomial can be written (in a unique way) as a sum of a monogenic polynomial and a products of powers of $\underline{x}$ with other monogenic polynomials.

2.

2.1. The deformation of the Fisher decomposition - scalar case

Let $r_\alpha, \alpha \in \mathbb{R}^n - \{0\}$ be the reflection in the hyperplane orthogonal to α. We shall consider a normalized root system in $\mathbb{R}^n$.

Definition 2.1. A finite set R of non-zero vectors is called a normalized root system, if $(\alpha, \alpha) = 2$ and $r_\alpha(R) = R$ for all $\alpha \in R$.

The Coxeter group G (corresponding to R) is a finite subgroup of the orthogonal group generated by the reflections $\{r_\alpha | \alpha \in R\}$.

The divided quotient Δ_α is a linear operator on $\mathcal{P}$, defined by

$$\Delta_\alpha(p)(x) = \frac{p(x) - p(r_\alpha(x))}{(\alpha, x)}.$$

A weight function k is a complex-valued function on R, which is constant on each orbit of W on R.

The vector space of all weight functions $k = k(\alpha)$ has the dimension equal to the number of W-orbits in R. A choice of k is a choice of a deformation. It is easy to check that the Δ_α map polynomials to polynomials. The Dunkl operators are deformations of partial derivatives, they depend on a choice of the weight function k (but not on a choice of the system R^+ of positive roots used in the definition below) and are defined as follows.

Definition 2.2. Let us choose the set R^+ of positive roots in R. For $\xi \in \mathbb{R}^n$, we shall denote derivative in a direction ξ by $\partial_\xi = \sum_j \xi_j \partial_j$. Recall the Dunkl operators T_ξ defined by

$$T_\xi := \partial_\xi + \sum_{\alpha \in R^+} k(\alpha) \Delta_\alpha < \xi, \alpha > .$$

Recall the basic properties of the Dunkl operators summarized in the following theorem.

Theorem 2.3.
(1) For all vectors $\xi, \eta \in \mathbb{R}^n$, the operators T_ξ, T_η commute.
(2) Let e_j be the canonical basis of $\mathbb{R}^n$ and let $T_j := T_{e_j}$. The Dunkl Laplacian Δ_k has the form

$$\Delta_k := \sum_j T_j^2 = \Delta + \sum_{\alpha \in R^+} k(\alpha) \frac{1}{< \alpha, . >} \{\partial_\alpha - \Delta_\alpha\}.$$

The classical realisation of $\mathrm{sl}(2, \mathbb{R})$ described in Sect.2.1 can be deformed to a realization generated by the Dunkl Laplacian and by the multiplication by $|x|^2$. Again, we have the important commutation relations:

Theorem 2.4. *Let $k \geq 0$. Let us denote*

$$E := |x|^2/2; \ F := \Delta_k/2; \ H := -(\mathbb{E} + n/2 + \gamma),$$

where $\mathbb{E} = \sum_i x_i \partial_i$, $\gamma = \sum_{\alpha \in R_+} k(\alpha)$.

Then

$$[H, E] = 2E,\ [H, F] = -2F,\ [E, F] = H.$$

Using it, we can apply the standard method to prove the deformed version of the Fisher decomposition - with $\tilde{\mathcal{H}}$ denoting the Dunkl harmonics. It has the same form of an infinite triangle.

$$\begin{array}{ccccccc}
\mathcal{P}_0 & \mathcal{P}_1 & \mathcal{P}_2 & \mathcal{P}_3 & \mathcal{P}_4 & \mathcal{P}_5 & \mathcal{P}_6 \\
\tilde{\mathcal{H}}_0 = \langle 1 \rangle & \stackrel{\Delta_k}{\longleftarrow} & \langle r^2 \rangle & \stackrel{\Delta_k}{\longleftarrow} & \langle r^4 \rangle & \stackrel{\Delta_k}{\longleftarrow} & \langle r^6 \rangle \\
 & \tilde{\mathcal{H}}_1 & \stackrel{\Delta_k}{\longleftarrow} & r^2\tilde{\mathcal{H}}_1 & \stackrel{\Delta_k}{\longleftarrow} & r^4\tilde{\mathcal{H}}_1 & \\
 & & \tilde{\mathcal{H}}_2 & \stackrel{\Delta_k}{\longleftarrow} & r^2\tilde{\mathcal{H}}_2 & \stackrel{\Delta_k}{\longleftarrow} & r^4\tilde{\mathcal{H}}_2 \\
 & & & \tilde{\mathcal{H}}_3 & \stackrel{\Delta_k}{\longleftarrow} & r^2\tilde{\mathcal{H}}_3 & \\
 & & & & \tilde{\mathcal{H}}_4 & \stackrel{\Delta_k}{\longleftarrow} & r^2\tilde{\mathcal{H}}_4
\end{array}$$

The first two rows do not change with a change of deformation. But the space $\tilde{\mathcal{H}}_2$ is changing its position among all possible subspaces of $\mathcal{P}_2(S)$, which are complementary to $|x|^2\tilde{\mathcal{H}}_0$. The same deformation is translated by the action of $|x|^2$ to other spaces in the third row.

3.

3.1. The deformation of the Fisher decomposition - spinor case

The discussion above shows that it is a natural idea to consider a Dunkl version of the Dirac operator, defined by

$$\underline{\mathcal{D}}_k := \sum_j e_j T_j.$$

The Dunkl Dirac operator has standard basic properties - it does not depend on a choice of an orthogonal basis used in its definition, is invariant under the action of the group G and its square is the Dunkl Laplace operator.

3.2. The Howe pair for the Dunkl Dirac operator

In this part, we shall compute certain anti- and commutation relations in the endomorphism ring $\mathrm{End}(\mathcal{P}(S))$. We shall show that the two natural operators $\underline{\mathcal{D}}_k$ and $\underline{x}$, considered as odd elements, generate a finite-dimensional Lie superalgebra in $\mathrm{End}(\mathcal{P}(S))$.

Lemma 3.1. *We have*

$$\{\underline{x}, \underline{x}\} = -2|x|^2; \quad \{\underline{\mathcal{D}}_k, \underline{\mathcal{D}}_k\} = -2\Delta_k;$$
$$\{\underline{x}, \underline{\mathcal{D}}_k\} = -2(\mathbb{E} + \frac{n}{2} + \gamma_k)$$

where $\gamma_k = \sum_{\alpha \in R^+} k(\alpha)$.

Proof.

The first relation is trivial. For the second one, it is sufficient to use commutativity of operators T_j to get

$$\{\underline{\mathcal{D}}_k, \underline{\mathcal{D}}_k\} = -2\sum_j (T_j)^2.$$

The key relation is the last one. We have

$$\underline{x\mathcal{D}_k} + \underline{\mathcal{D}_k x} =$$

$$\sum_{i,j;i\neq j} e_je_i(x_jT_i + T_jx_i) - \sum_i (x_iT_i + T_ix_i).$$

We shall show first that the term

$$\sum_{i,j;i\neq j} e_je_i(x_jT_i + T_jx_i)$$

vanishes. Due to

$$\sum_{i,j;i\neq j} e_je_i(x_jT_i + x_iT_j) = 0,$$

we have to show $\sum_{i,j;i\neq j} e_je_i(T_jx_i - x_iT_j) = 0$. But for all smooth functions f

$$T_j(x_if) - x_iT_jf =$$

$$\sum_{\alpha\in R^+} k(\alpha)\frac{x_if(x) - (r_\alpha x)_if(r_\alpha x)}{< x, \alpha >}(\alpha)_j$$

$$-x_i\left(\sum_{\alpha\in R^+} k(\alpha)\frac{f(x) - f(r_\alpha x)}{< x, \alpha >}(\alpha)_j\right) =$$

$$2\sum_{\alpha\in R^+} k(\alpha)\frac{(\alpha)_i(\alpha)_jf(r_\alpha x)}{< x, \alpha >},$$

with an expression obviously symmetric in i, j.

The second sum on the right hand side of the above gives

$$\sum_i (T_i(x_i f) + x_i T_i f) =$$

$$\sum_i \sum_{\alpha \in R^+} k(\alpha) \frac{x_i f(x) - (r_\alpha x)_i f(r_\alpha x)}{< x, \alpha >} (\alpha)_i$$

$$+ \sum_i \sum_{\alpha \in R^+} k(\alpha) x_i \frac{f(x) - f(r_\alpha x)}{< x, \alpha >} (\alpha)_i =$$

$$2f(x) \sum_{\alpha \in R^+} k(\alpha),$$

because $< r_\alpha x, \alpha >= - < x, \alpha >$. It finishes the proof.

Lemma 3.2. *The operators* $\underline{x}$ *and* $\underline{\mathcal{D}}_k$ *generate a finite-dimensional Lie superalgebra in* $\mathrm{End}(\mathcal{P}(S)$. *The remaining commutation relations are*

$[\underline{x}, |x|^2] = 0;\ [\underline{x}, \Delta_k] = -2\underline{\mathcal{D}}_k;\ [\underline{x}, \mathbb{E}] = -\underline{x};$
$[\underline{\mathcal{D}}_k, |x|^2] = 2\underline{x};\ [\underline{\mathcal{D}}_k, \Delta_k] = 0;\ [\underline{\mathcal{D}}_k, \mathbb{E}] = \underline{\mathcal{D}}_k;$

$$[|x|^2, \Delta_k] = -4(\mathbb{E} + \frac{m}{2} + \gamma);$$
$$[\mathbb{E}, |x|^2] = 2|x|^2;\ [\mathbb{E}, \Delta_k] = -2\Delta_k.$$

Proof. The relations $[\underline{x}, |x|^2] = 0 = [\underline{\mathcal{D}}_k, \Delta_k]$ are trivial (T_j and T_k commmute). Four commutation relations containing the Euler operator $\mathbb{E}$ are immediate. The function $|x|^2$ is G-invariant, hence for all polynomials $P \in \mathcal{P}(S)$
$$T_j(|x|^2\, p) = T_j(|x|^2)\, p + |x|^2 T_j(p).$$
This implies $[\underline{\mathcal{D}}_k, |x|^2] = 2\underline{x}$.

The relation $[x_i, \Delta_k] = -2T_i$ we get as before, hence $[\underline{x}, \Delta_k] = -2\underline{\mathcal{D}}$. Finally, we have $[|x|^2, \Delta_k] = -4(\mathbb{E} + \frac{m}{2} + \gamma)$ which is also easy to deduce from $\{\underline{x}, \underline{\mathcal{D}}_k\} = -2(\mathbb{E} + \frac{n}{2} + \gamma_k)$ using a simple operator identity
$$[A^2, B^2] = \{[A, \{A, B\}], B\}$$
and other known anti- and commutation relations.

The set of relations in Lemma 3.2 shows that we have got a finite dimensional super

Lie algebra, isomorphic to osp(1|2). Indeed, the normalization

$$H = \frac{1}{2}(\mathbb{E}+n+\gamma), \quad E^+ = \frac{1}{2}|x|^2, \quad E^- = -\frac{1}{2}\Delta_k$$

and

$$F^+ = \frac{1}{2\sqrt{2}}\, i\underline{x}, \quad F^- = \frac{1}{2\sqrt{2}}\, i\underline{\mathcal{D}}_k$$

leads the standard commutation relations for osp(1|2)

$$\begin{array}{rclrcl} [H, E^{\pm}] &=& \pm E^{\pm}; & [E^+, E^-] &=& 2H; \\ [H, F^{\pm}] &=& \pm\frac{1}{2}F^{\pm}; & \{F^+, F^-\} &=& \frac{1}{2}H; \\ [E^{\pm}, F^{\mp}] &=& -F^{\pm}; & \{F^{\pm}, F^{\pm}\} &=& \pm\frac{1}{2}E^{\pm}. \end{array}$$

3.3. Separation of variables and the Fisher decomposition

Having the deformation of the Howe dual pair at hand, it is now a standard procedure to describe the Dunkl deformation of the classical separation of variables.

Theorem 3.3. *Let $\mathcal{M}_k$ denote the space of all k-homogeneous polynomials in the kernel of the Dunkl Dirac operator $\underline{\mathcal{D}}_k$.*

The space $\mathcal{P}(S)$ decomposes as a $G \times$ $\mathrm{osp}(1|2)$-module to the direct sum

$$\oplus_{k=0}^{\infty} \oplus_{j=0}^{\infty} \underline{x}^j \mathcal{M}_k].$$

The decomposition has a form of an infinite triangle

$$\begin{array}{cccccccc}
\mathcal{P}_0(S) & & \mathcal{P}_1(S) & & \mathcal{P}_2(S) & & \mathcal{P}_3(S) & \\
 & & & & & & & \\
\mathcal{M}_0 = \langle 1 \rangle & \xleftarrow{D} & \underline{x}\mathcal{M}_0 & \xleftarrow{D} & |x|^2\mathcal{M}_0 & \xleftarrow{D} & \underline{x}^3\mathcal{M}_0 & \dots \\
 & & \oplus & & \oplus & & \oplus & \\
 & & \mathcal{M}_1 & \xleftarrow{D} & \underline{x}\mathcal{M}_1 & \xleftarrow{D} & |x|^2\mathcal{M}_1 & \dots \\
 & & & & \oplus & & \oplus & \\
 & & & & \mathcal{M}_2 & \xleftarrow{D} & \underline{x}\mathcal{M}_2 & \dots \\
 & & & & & & \oplus & \\
 & & & & & & \mathcal{M}_3 & \dots
\end{array}$$

where all summands are G-invariant (not irreducible, in general) and all rows are $\mathrm{osp}(1|2)$*-invariant.*

Proof. The commutation relation for $\mathrm{osp}(1|2)$ imply that the rows in the triangle are invariant subspaces. The multiplication by $|x|^2$

acts as an isomorphism between individual summands in a row. The same is true for the Dunkl Dirac operator $\underline{\mathcal{D}}_k$ (with exception of the first summand $\mathcal{M}_k$).

Consequently, we get the following Fisher decomposition.

Theorem 3.4. *We have*

$$\mathcal{P}_j(S) = \oplus_{i=0}^{j} \underline{x}^i \mathcal{M}_{j-i}.$$

3.4. Explicit formulae

We shall now give explicit formulae for the Fisher decomposition.

Lemma 3.5. *For any $p \in \mathcal{P}_j(S)$ and any positive integer s, the following relations hold:*

$$\underline{\mathcal{D}}\, \underline{x}^s\, p = d_{s,j} \underline{x}^{s-1}\, p + (-1)^s \underline{x}^s \underline{\mathcal{D}}\, p,$$

where $d_{2t,j} = -2t$; $d_{2t+1,j} = \alpha_{j+t}$; $\alpha_u = -2(\frac{m}{2} + \gamma_k + u)$.

Proof. It is sufficient to commute $\underline{\mathcal{D}}$ through $\underline{x}^s$ using the relation $\underline{\mathcal{D}}\, \underline{x} + \underline{x}\, \underline{\mathcal{D}} = -2(\mathbb{E} +$

$\frac{m}{2}+\gamma_k)$. The resulting coefficients are equal to

$$d_{2t,j} = \alpha_{j+2t-1} - \alpha_{j+2t-2} + \alpha_{j+2t-3} - \ldots + \alpha_{j+1} - \alpha_j$$
$$= -2t;$$
$$d_{2t+1,j} = \alpha_{j+2t} - \alpha_{j+2t-1} + \alpha_{j+2t-2} - \alpha_{j+2t-3} + \ldots - \alpha_{j+1} + \alpha_j$$
$$= \alpha_{j+t}.$$

Let us now compute an explicit form of the projection $\pi_{\mathcal{M}}$ from $\mathcal{P}_j(S)$ to $\mathcal{M}_j$.

Lemma 3.6. *Let $(a)_s = a(a+1)\ldots(a+s-1)$ denote the Pochhammer symbol. We shall define constants $c_{s,j}$ by*

$$c_{2s,j} = \frac{(-1)^s}{4^s\, s!\, (\frac{m}{2}+\gamma_k+j-s)_s}$$

$$c_{2s+1,j} = \frac{(-1)^s}{2\cdot 4^s\, s!\, (\frac{m}{2}+\gamma_k+j-s-1)_{s+1}};\ c_{0,j} = 1.$$

Then the map $\pi_{\mathcal{M}}$ given by

$$\pi_{\mathcal{M}}(p) := p + c_{1,j}\underline{x}\,\underline{\mathcal{D}}\,p + c_{2,j}\underline{x}^2\underline{\mathcal{D}}^2\,p + \ldots + c_{j,j}\underline{x}^j\underline{\mathcal{D}}^j\,p$$

is the projection from $\mathcal{P}_j(S)$ to $\mathcal{M}_j$.

Proof. Let us consider a linear combination $r := a_0 p + a_1 \underline{x}\,\underline{\mathcal{D}}\, p + a_2 \underline{x}^2 \underline{\mathcal{D}}^2\, p + \ldots + a_j \underline{x}^j \underline{\mathcal{D}}^j\, p$ with $a_0 = 1$. If there are constants a_j such that $r \in \mathcal{M}_j$, then r is equal to $\pi_{\mathcal{M}}(p)$. Indeed, we know that $\mathcal{P}_j(S) = \mathcal{M}_j \oplus \underline{x}\, \mathcal{P}_{j-1}(S)$ and

$$r = p + q, q = \sum_{i=1}^{j} a_i \underline{x}^i \underline{\mathcal{D}}^i\, p \in \underline{x} \mathcal{P}_{j-1}(S).$$

Applying Lemma 3.5, we get

$$0 = \underline{\mathcal{D}}(\pi_{\mathcal{M}}(p)) = \underline{\mathcal{D}} p + a_1 \underline{\mathcal{D} x \mathcal{D}} p + a_2 \underline{\mathcal{D} x}^2 \underline{\mathcal{D}}^2 p + \ldots + a_j \underline{\mathcal{D} x}^j \underline{\mathcal{D}}^j p =$$

$$\underline{\mathcal{D}} p + a_1 d_{1,j-1} \underline{\mathcal{D}} p - a_1 \underline{x \mathcal{D}}^2 p + a_2 d_{2,j-2} \underline{x \mathcal{D}}^2 p + a_2 \underline{x}^2 \underline{\mathcal{D}}^3 p + a_3 d_{3,j-3} \underline{x}^2 \underline{\mathcal{D}}^3 p - a_3 \underline{x}^3 \underline{\mathcal{D}}^4 p + \ldots + (-1)^{j-1} a_{j-1} \underline{x}^{j-1} \underline{\mathcal{D}}^j p + a_j d_{j,0} \underline{x}^{j-1} \underline{\mathcal{D}}^j p.$$

Hence if the relation $(-1)^{i-1} a_{i-1} + a_i d_{i,j-i} = 0$ hold for each $i = 1, \ldots, j$, the function r is monogenic.

By induction, we get

$$a_i = \frac{(-1)^{\lfloor \frac{i}{2} \rfloor}}{(-d_{1,j-1}) \dots (-d_{i,j-i})}.$$

The lemma then follows from

$$-d_{1,j-1} = 2\left(\frac{m}{2} + \gamma_k + j - 1\right), \quad -d_{2,j-2} = 2,$$
$$-d_{3,j-3} = 2\left(\frac{m}{2} + \gamma_k + j - 2\right), \quad -d_{4,j-4} = 4, \dots$$

Theorem 3.7. *Each polynomial $p \in \mathcal{P}_j$ can be written in a unique way as*

$$p = \sum_{\ell=0}^{j} \underline{x}^{\ell} M_{j-\ell}, \; M_{j-\ell} = M_{j-\ell}(p) \in \mathcal{M}_{j-\ell}.$$

The individual components in the decomposition are given by

$$M_{j-\ell}(p) = c'_\ell \sum_{n=0}^{j-\ell} c_{n,j-\ell} \underline{x}^n \underline{\mathcal{D}}^{n+\ell} p; \; \ell = 0, \dots, j,$$

where c'_ℓ are defined by

$$c'_0 = 1;\ c'_{2s} = \frac{1}{4^s (s)! (\frac{m}{2} + \gamma_k + j - 2s)_s},\ s > 0;$$
$$c'_{2s+1} = \frac{-1}{2 \cdot 4^s (s)! (\frac{m}{2} + \gamma_k + j - 2s - 1)_{s+1}},\ s \geq 0.$$

Proof. We know that for any $p \in \mathcal{P}_j(S)$, there is a unique decomposition

$$p = \underline{x}^j M_0 + \underline{x}^{j-1} M_1 + \ldots + \underline{x} M_{j-1} + M_j,$$
$$M_{j-\ell} \in \mathcal{M}_{j-\ell},\ \ell = 0, \ldots, j.$$

To compute a component $M_{j-\ell}$ explicitly, we apply $\underline{\mathcal{D}}^\ell$ to both sides of the relation:

$$\underline{\mathcal{D}}^\ell p = \underline{\mathcal{D}}^\ell \underline{x}^j M_0 + \underline{\mathcal{D}}^\ell x^{j-1} M_1$$
$$+ \ldots + \underline{\mathcal{D}}^\ell \underline{x}^{\ell+1} M_{j-\ell-1} + \underline{\mathcal{D}}^\ell \underline{x}^\ell M_{j-\ell}.$$

The summands on the right hand side belong, in turn, to the spaces

$$\underline{x}^{j-\ell} \mathcal{M}_0, \underline{x}^{j-\ell-1} \mathcal{M}_1, \ldots, \underline{x} \mathcal{M}_{j-\ell-1}, \mathcal{M}_{j-\ell}.$$

Hence $\underline{\mathcal{D}}^\ell \underline{x}^\ell M_{j-\ell}$ is equal to $\pi_{\mathcal{M}}(\underline{\mathcal{D}}^\ell p)$. We can now use the expression for the harmonic projection proved above. So to get the statement of the theorem, it is sufficient to show that

$$\underline{\mathcal{D}}^{\ell}\underline{x}^{\ell}M_{j-\ell} = \frac{1}{c'_\ell}M_\ell.$$

Using Lemma 3.5, we get by induction

$$c'_\ell = \frac{1}{d_{1,j-\ell}\dots d_{\ell,j-\ell}}.$$

This finishes the proof of the theorem.