

**Aim and background.** One important aspect of Dunkl analysis is to consider the role of the Dunkl Laplacian, deforming the usual Laplacian by a parameter  $k$ . Several people have worked on this, viz.

- K. Trimèche
- M. Sifi, F. Soltani
- S. Ben Said
- M. Rösler
- M. F. de Jeu
- E. Opdam
- G. J. Heckman
- C. Dunkl and Y. Xu

to name a few. Among the topics studied have been

- properties of special functions
- the deformation of the heat kernel
- intertwining operators

- Dunkl's deformation of the Fourier transform
- analogues of the Segal-Bargmann transform
- analogues of the wave equation
- further deformations of the Fourier transform

After introducing the basic material, these lectures will first investigate a generalization of both the Fock spaces and the Segal-Bargmann transform in the setting of Coxeter groups and Dunkl operators. The motivation for studying the Segal-Bargmann transform is to exhibit some relationships between Dunkl's theory and its applications in the Schrödinger model and in the Fock model which includes the study of the Calogero-Moser systems, and the Dunkl transform. We will also prove that one can develop an analogous theory of Howe dual pairs to obtain the branching decomposition of the generalized Fock space. It turns out, as a

key observation, that there exists an  $\mathfrak{sl}(2)$ -triple based on the Dunkl-Laplacian (see below) giving rise to an interesting unitary representation of the universal covering  $\widetilde{SL}(2, \mathbb{R})$ , that is an analogue to the classical metaplectic representation of  $Mp(2, \mathbb{R})$ . By means of this representation, we may also prove a Bochner formula for the Dunkl transform. Furthermore, this allows to investigate the validity of Huygens' principle for wave equations for the Dunkl-Laplacian operators. To do so, we adapt R. Howe's method for the Euclidean Fourier transform, and for the classical wave equation, respectively.

## 1. BASIC NOTIONS

Let  $\langle \cdot, \cdot \rangle$  be the standard Euclidean scalar product in  $\mathbb{R}^N$ , as well as its bilinear extension to  $\mathbb{C}^N \times \mathbb{C}^N$ . For  $\alpha \in \mathbb{R}^N \setminus \{0\}$ , let  $r_\alpha$  be the reflection on the hyperplane

$\langle \alpha \rangle^\perp$  orthogonal to  $\alpha$

$$r_\alpha(x) := x - 2 \frac{\langle \alpha, x \rangle}{\langle \alpha, \alpha \rangle} \alpha, \quad x \in \mathbb{R}^N.$$

Let  $\mathcal{R}$  be a reduced root system, i.e.  $\mathcal{R} \cap \mathbb{R}\alpha = \{\pm\alpha\}$  for all  $\alpha \in \mathcal{R}$  and  $r_\alpha(\mathcal{R}) = \mathcal{R}$ . Henceforth, we will normalize  $\mathcal{R}$  in the sense that  $\langle \alpha, \alpha \rangle = 2$ . This simplifies formulas, with no loss of generality for our purposes.

A Coxeter group  $G$  is a finite subgroup of the orthogonal group  $O(N)$  generated by the reflections  $\{r_\alpha \mid \alpha \in \mathcal{R}\}$ . Note that Coxeter groups generalize Weyl groups since there is no additional crystallographic condition for  $\mathcal{R}$ .

A multiplicity function on  $\mathcal{R}$  is a  $G$ -invariant function  $k : \mathcal{R} \rightarrow \mathbb{C}$ . We set  $\mathcal{K}^+$  to be the set of multiplicity functions  $k = (k_\alpha)_{\alpha \in \mathcal{R}}$  such that  $k_\alpha \geq 0$  for all  $\alpha$ , and we let  $\mathcal{R}^+$  be a choice of positive roots in  $\mathcal{R}$ .

Around 1990, C. Dunkl defined a family of first order differential-difference operators

that play the role of the usual partial differentiation. Dunkl's operators are defined by

$$T_{\xi}(k)f(x) = \partial_{\xi}f(x) + \sum_{\alpha \in \mathcal{R}^+} k_{\alpha} \langle \alpha, \xi \rangle \frac{f(x) - f(r_{\alpha}x)}{\langle \alpha, x \rangle}, \quad f \in \mathcal{C}^1(\mathbb{R}^N),$$

where  $\partial_{\xi}$  denotes the directional derivative corresponding to  $\xi$ . In particular, for any orthonormal basis  $\{\xi_i\}_{i=1}^N$  of  $\mathbb{R}^N$ , the Dunkl-Laplacian operator  $\Delta_k := \sum_{i=1}^N T_{\xi_i}^2(k)$  can be written as

$$\Delta_k f(x) = \Delta f(x) + 2 \sum_{\alpha \in \mathcal{R}^+} k_{\alpha} \left\{ \frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(r_{\alpha}x)}{\langle \alpha, x \rangle^2} \right\},$$

where  $\Delta$  and  $\nabla$  denote the usual Laplacian and gradient, respectively. For all  $i$ -th basis vectors  $\xi_i$ , we will use the abbreviation  $T_{\xi_i}(k) = T_i(k)$ .

For  $k \in \mathcal{K}^+$ , there exists a generalization of the usual exponential kernel  $e^{\langle \cdot, \cdot \rangle}$

by means of the Dunkl system of differential equations.

**Theorem 1.1.** *For  $k \in \mathcal{K}^+$ , there exists a unique function  $E_k$  on  $\mathbb{C}^N \times \mathbb{C}^N$  characterized by:*

- (i)  $T_\xi(k)E_k(z, w) = \langle \xi, w \rangle E_k(z, w)$ ; and
- (ii)  $E_k(z, 0) = 1$ .

*Moreover, this function satisfies*

- (iii)  $E_k$  is holomorphic on  $\mathbb{C}^N \times \mathbb{C}^N$ ;
- and*
- (iv)  $E_k(g_0 \cdot z, g_0 \cdot w) = E_k(z, w)$  for all  $g_0 \in G$ .

For complex-valued  $k$ , there is a detailed investigation of (i) by Opdam. Theorem 2.1 is a weak version of Opdam's result. The function  $E_k$  is the so-called Dunkl kernel. When  $k \equiv 0$ , we have  $E_0(z, w) = e^{\langle z, w \rangle}$  for  $z, w \in \mathbb{C}^N$ .

For  $k \in \mathcal{K}^+$ , let  $\omega_k$  be the weight function on  $\mathbb{R}^N$  defined by

$$\omega_k(x) := \prod_{\alpha \in \mathcal{R}^+} |\langle \alpha, x \rangle|^{2k_\alpha}.$$

Further, let

$$c_k := \int_{\mathbb{R}^N} e^{-\langle x, x \rangle/2} \omega_k(x) dx,$$

which is called the Macdonald-Metha-Selberg integral. The following proposition is crucial in Dunkl's theory and its applications.

**Proposition 1.2.** *Let  $z, w \in \mathbb{C}^N$ . For non-negative multiplicity function  $k$ ,*

$$\int_{\mathbb{R}^N} E_k(x, z) E_k(x, w) e^{-\langle x, x \rangle/2} \omega_k(x) dx = c_k e^{(\langle z, z \rangle + \langle w, w \rangle)/2} E_k(z, w).$$

For  $z, w \in \mathbb{C}^N$ , define

$$\mathbb{K}_{k,w}(z) = \mathbb{K}_k(z, w) := E_k(z, \bar{w}).$$

As  $k$  will be fixed, we will write  $\mathbb{K}$  for  $\mathbb{K}_k$ . By Theorem 2.1, one may check that  $\mathbb{K}$  is continuous and  $\mathbb{K}_w$  is holomorphic for all

$w \in \mathbb{C}^N$ . Further,  $\mathbb{K}(z, w) = \overline{\mathbb{K}(w, z)}$ . Another crucial property is that  $\mathbb{K}(z, w)$  is a positive definite kernel, i.e. for all  $z^{(1)}, \dots, z^{(\ell)} \in \mathbb{C}^N$  and  $a_1, \dots, a_\ell \in \mathbb{C}$

$$\sum_{i,j=1}^{\ell} a_i \overline{a_j} \mathbb{K}(z^{(i)}, z^{(j)}) \geq 0.$$

These properties of  $\mathbb{K}$  lead to the following result.

**Theorem 1.3.** (i) *There exists a Hilbert space  $\mathcal{F}_k(\mathbb{C}^N)$  of holomorphic functions on  $\mathbb{C}^N$ , such that  $\mathbb{K}$  is its reproducing kernel.*

(ii) *The Hilbert space  $\mathcal{F}_k(\mathbb{C}^N)$  contains the  $\mathbb{C}$ -algebra  $\mathcal{P}(\mathbb{C}^N)$  of polynomial functions on  $\mathbb{C}^N$  as a dense subspace.*

Now we have the integral representation of the Segal-Bargmann transform  $\mathcal{B}_k$  given by

**Theorem 1.4.** *A unitary isomorphism  $\mathcal{B}_k : \mathcal{L}^2(\mathbb{R}^N, \omega_k) \rightarrow \mathcal{F}_k(\mathbb{C}^N)$  is defined*

by

$$\mathcal{B}_k f(z) = 2^{\gamma_k + N/2} c_k^{-1/2} e^{-\langle z, z \rangle / 2}$$

$$\int_{\mathbb{R}^N} f(x) E_k(\sqrt{2}x, \sqrt{2}z) e^{-\langle x, x \rangle} \omega_k(x) dx,$$

where  $\gamma_k := \sum_{\alpha \in \mathcal{R}^+} k_\alpha$ .

## 2. LIST OF TOPICS

The aim of the lectures is to first present the details of some of the results above; depending on the interests of the audience more applications of the basic functions and the  $\mathfrak{sl}(2, \mathbb{R})$ -representation discussed above may be given. A possible outline of topics is:

- Construction of the generalized Segal-Bargmann transform
- Dual pairs associated with Dunkl operators
- Wave equations for Dunkl differential-difference operators and Huygens' principle

- The Dunkl transform and Bochner-type formulas
- Bessel functions and analysis on flat symmetric spaces
- Further deformations of the Fourier transform
- The Dunkl operator analogue of the Dirac operator

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