1. Fock spaces and Segal-Bargmann transforms

Recall that we have a natural Fock space associated a root system:

Theorem 1.1. (i) *There exists a Hilbert space* $\mathscr{F}_k(\mathbb{C}^N)$ *of holomorphic functions on* $\mathbb{C}^N$*, such that* $\mathbb{K}$ *is its reproducing kernel.*

(ii) *The Hilbert space* $\mathscr{F}_k(\mathbb{C}^N)$ *contains the* $\mathbb{C}$*-algebra* $\mathscr{P}(\mathbb{C}^N)$ *of polynomial functions on* $\mathbb{C}^N$ *as a dense subspace.*

In particular, if we denote by $\langle\!\langle \cdot, \cdot \rangle\!\rangle_k$ the inner product in $\mathscr{F}_k(\mathbb{C}^N)$, then

$$\langle\!\langle p, q \rangle\!\rangle_k = p(T(k))\overline{q(\bar{z})}\Big|_{z=0}, \qquad \forall\, p, q \in \mathscr{P}(\mathbb{C}^N),$$

where $p(T(k))$ is the operator formed by replacing z_i by $T_i(z)$ for $1 \le i \le N$.

If $k \equiv 0$, $\mathscr{F}_0(\mathbb{C}^N)$ coincides with the classical Fock space. We shall call $\mathscr{F}_k(\mathbb{C}^N)$ the Fock space associated with the Coxeter G.

***Example* 1.2.** Let $\{e_1, \ldots, e_N\}$ be the standard basis of $\mathbb{R}^N$, and consider the reflection group G generated by the reflections $r_1, \ldots, r_N$ along $e_1, \ldots, e_N$, i.e.

$$r_j(\cdots, x_{j-1}, x_j, x_{j+1}, \cdots) =$$

$$(\cdots, x_{j-1}, -x_j, x_{j+1}, \cdots)$$

for $x \in \mathbb{R}^N$.

For a multi-parameter $k = (k_1, \ldots, k_N)$ such that $k_i \geq 0$, the Dunkl operators take the form

$$T_j(k)f(x) = \partial_j f(x) +$$

$$k_j \frac{f(x) - f(r_j x)}{x_j}, \qquad 1 \leq j \leq N, \ x \in \mathbb{R}^N$$

The Dunkl kernel E_k is given by

$$E_k(z, w) = \prod_{j=1}^{N} \Gamma\Big(k_j + \frac{1}{2}\Big)\Big(\frac{z_j w_j}{2}\Big)^{1/2 - k_j}$$

$$\{I_{k_j-1/2}(z_jw_j)+I_{k_j+1/2}(z_jw_j)\},$$

where I_ν is the modified Bessel function of the first kind. In this example, the Fock space $\mathscr{F}_k(\mathbb{C}^N)$ related to $G\cong(\mathbb{Z}/2\mathbb{Z})^N$ is the Hilbert space of holomorphic functions on $\mathbb{C}^N$ which are square integrable with respect to the measure

$$d\mu_k(z)=\prod_{j=1}^N\frac{|z_j|^{2k_j+1}}{\pi 2^{k_j-1/2}\Gamma(k_j+1/2)}\Big\{\mathscr{K}_{k_j-1/2}(|z_j|^2)\Big|_{\text{even part}}+\mathscr{K}_{k_j+1/2}(|z_j|^2)\Big|_{\text{odd part}}\Big\}dz_j,$$

splitting functions into even and odd parts in each variable z_j. Here $\mathscr{K}_\nu$ is the Bessel function of the third kind.

For $t>0$ and $z,w\in\mathbb{C}^N$, set

$$\Gamma_k(t,z,w)=\frac{1}{(2t)^{\gamma_k+N/2}c_k}e^{-(\langle z,z\rangle+\langle w,w\rangle)/4t}E_k\Big(\frac{z}{\sqrt{2t}},\frac{w}{\sqrt{2t}}\Big).$$

The kernel $\Gamma_k(t,z,w)$ was introduced by Rösler as a generalized heat kernel.

Let $\mathscr{L}^2(\mathbb{R}^N,\omega_k)$ be the space of $\mathscr{L}^2$-functions on $\mathbb{R}^N$ with respect to the weight function ω_k.

The restriction principle.

Let $\mathcal{R}_k$ be the restriction map $\mathcal{R}_k : \mathscr{F}_k(\mathbb{C}^N) \to \mathscr{L}^2(\mathbb{R}^N,\omega_k)$, given by

$$\mathcal{R}_k f(x) := e^{-\langle x,x\rangle/2} f(x), \quad x \in \mathbb{R}^N.$$

The map $\mathcal{R}_k$ is a closed, densely defined operator from $\mathscr{F}_k(\mathbb{C}^N)$ into $\mathscr{L}^2(\mathbb{R}^N,\omega_k)$ with dense image (see for instance [**?**, Corollary 3.5]). Consider the adjoint $\mathcal{R}_k^* : \mathscr{L}^2(\mathbb{R}^N,\omega_k) \to \mathscr{F}_k(\mathbb{C}^N)$ as a densely defined operator. Since $\mathbb{K}$ is the reproducing kernel of $\mathscr{F}_k(\mathbb{C}^N)$, one can prove that for $f \in \mathscr{L}^2(\mathbb{R}^N,\omega_k)$, the integral

$$\mathcal{R}_k\mathcal{R}_k^* f(y) = c_k \int_{\mathbb{R}^N} f(x)\Gamma_k\Big(\frac{1}{2},x,y\Big)\omega_k(x)\,dx$$

converges absolutely for a.e. $y \in \mathbb{R}^N$. The function $\mathcal{R}_k\mathcal{R}_k^* f$ thus defined is in $\mathscr{L}^2(\mathbb{R}^N,\omega_k)$

and $\|\mathcal{R}_k\mathcal{R}_k^*\| \le c_k$. We can therefore define $\sqrt{\mathcal{R}_k\mathcal{R}_k^*}$ and there exists an isometry $\mathscr{B}_k$ so that $\mathcal{R}_k^* = \mathscr{B}_k\sqrt{\mathcal{R}_k\mathcal{R}_k^*}$. Since $\mathcal{R}_k = \sqrt{\mathcal{R}_k\mathcal{R}_k^*}\mathscr{B}_k^*$ and Image$(\mathcal{R}_k)$ is dense, it follows that $\mathscr{B}_k$ is a unitary isomorphism. We shall call $\mathscr{B}_k$ the Segal-Bargmann transform associated with G. Using the positivity of the heat kernel $\Gamma(t,x,y)$ as an operator [**?**], we obtain the following integral representation of the Segal-Bargmann transform $\mathscr{B}_k$.

Theorem 1.3. *The unitary isomorphism* $\mathscr{B}_k : \mathscr{L}^2(\mathbb{R}^N, \omega_k) \to \mathscr{F}_k(\mathbb{C}^N)$ *is given by*

$$\mathscr{B}_k f(z) = 2^{\gamma_k+N/2} c_k^{-1/2} e^{-\langle z,z\rangle/2}$$

$$\int_{\mathbb{R}^N} f(x) E_k(\sqrt{2}x, \sqrt{2}z) e^{-\langle x,x\rangle} \omega_k(x)\, dx,$$

where $\gamma_k := \sum_{\alpha\in\mathscr{R}^+} k_\alpha$.

***Remark* 1.4.** (i) For the special case $k \equiv 0$,

$$\mathscr{B}_0 f(z) = (2/\pi)^{N/4}$$

$$\int_{\mathbb{R}^N} e^{-\langle x,x\rangle+2\langle x,z\rangle-\langle z,z\rangle/2} f(x)\,dx.$$

This compares well with the classical Segal-Bargmann transform (cf. [**?**, p. 40]).

(ii) As an infinite-order differential operator

$$\mathscr{B}_k^{-1} = 2^{\gamma_k+N/2} c_k^{-1/2} e^{\langle\,\cdot\,,\,\cdot\,\rangle}\; d_2 \circ e^{-\Delta_k/2}, \tag{1.1}$$

where d_2 is the dilation operator on functions by 2.

The Dunkl transform shares many properties with the Euclidean Fourier transform. further studied in [**?**].We will write the Dunkl transform as

$$\mathscr{D}_k f(\xi) = c_k^{-1} 2^{-\gamma_k-N/2}$$

$$\int_{\mathbb{R}^N} f(x/2) E_k(-i\xi, x)\omega_k(x)\,dx, \quad \xi \in \mathbb{R}^N.$$

Theorem 1.5. *The following diagram commutes*

$$\begin{array}{ccc} \mathscr{L}^2(\mathbb{R}^N,\omega_k) & \xrightarrow{\mathscr{B}_k} & \mathscr{F}_k(\mathbb{C}^N) \\ \mathscr{D}_k \big\downarrow & & \big\downarrow (-i)^* \\ \mathscr{L}^2(\mathbb{R}^N,\omega_k) & \xrightarrow{\mathscr{B}_k} & \mathscr{F}_k(\mathbb{C}^N) \end{array}$$

where $(-i)^*f(z) := f(-iz)$ *for* $f \in \mathscr{F}_k(\mathbb{C}^N)$.

For $\xi \in \mathbb{C}^N$, denote by M_ξ the operator $M_\xi(f)(z) := \langle z, \xi\rangle f(z)$. Define the lowering and the raising operators on $\mathscr{L}^2(\mathbb{R}^N,\omega_k)$ by

$$A_\xi^- := \frac{1}{\sqrt{2}}(M_{2\xi} + T_\xi(k))$$
$$A_\xi^+ := \frac{1}{\sqrt{2}}(M_{2\xi} - T_\xi(k)).$$

These two operators were introduced by Rösler in connection with Rodrigues-type formulas for the eigenfunctions of the Calogero-Moser systems. Next we will see that these two operators, in the Fock model, are also the lowering and the raising operators on $\mathscr{F}_k(\mathbb{C}^N)$ in a more natural way.

Below, we will exhibit some relationships between operators on $\mathscr{L}^2(\mathbb{R}^N, \omega_k)$ and on $\mathscr{F}_k(\mathbb{C}^N)$. For an operator $\mathscr{O}$ on $\mathscr{L}^2(\mathbb{R}^N, \omega_k)$, we define the operator $\widetilde{\mathscr{O}}$ on $\mathscr{F}_k(\mathbb{C}^N)$ by

$$\widetilde{\mathscr{O}} = \mathscr{B}_k \circ \mathscr{O} \circ \mathscr{B}_k^{-1}.$$

Further, as usual, $[A, B] = AB - BA$ for $A, B \in \mathrm{End}(\mathscr{P}(\mathbb{C}^N))$.

Theorem 1.6. *The following properties hold:*

(i) $\widetilde{T_\xi}(k) = T_\xi(k) - M_\xi$ *for* $\xi \in \mathbb{C}^N$;
(ii) $[\widetilde{T_\xi}(k), \widetilde{T_\eta}(k)] = 0$ *for* $\xi, \eta \in \mathbb{C}^N$;
(iii) $\widetilde{M_{2\xi}} = T_\xi(k) + M_\xi$ *for* $\xi \in \mathbb{C}^N$;
(vi) $[\widetilde{M_{2\xi}}, \widetilde{M_{2\eta}}] = 0$ *for* $\xi, \eta \in \mathbb{C}^N$;
(v) $[\widetilde{T_\xi}(k), \widetilde{M_{2\eta}}] = 2\langle \xi, \eta\rangle + 2\sum_{\alpha\in\mathscr{R}^+} k_\alpha \langle \alpha, \xi\rangle\langle \alpha, \eta\rangle r_\alpha$; *and*
(vi) $\widetilde{A_\xi^-} = \sqrt{2}T_\xi(k)$, *and* $\widetilde{A_\xi^+} = \sqrt{2}M_\xi$.

Notice that, as the Dunkl operators are homogeneous of degree -1 *on polynomials, and since* M_ξ *are the multiplication operators, now obviously* $\widetilde{A_\xi^-}$ *and* $\widetilde{A_\xi^+}$ *are* *the lowering and the raising operators on* $\mathscr{P}(\mathbb{C}^N)$.

The above theorem, which is of independent interest, is mainly useful to obtain the quantum Calogero-Moser (CM) rational system in the Fock model. Let

$$\mathscr{L}_k := \Delta - 2 \sum_{\alpha \in \mathscr{R}^+} \frac{1}{\langle \alpha, x \rangle^2} k_\alpha (k_\alpha - r_\alpha),$$

and consider the following gauge equivalent version

$$\begin{aligned} \mathscr{H}_k &:= \tfrac{1}{4} \omega_k^{-1/2} (-\mathscr{L}_k + 4\langle x, x \rangle) \omega_k^{1/2} \\ &= \tfrac{1}{4}(-\Delta_k + 4\langle x, x \rangle) \end{aligned}$$

of the CM Hamiltonian with harmonic confinement and reflection terms. These operators are introduced by Heckman [**?**] to prove the quantum integrability of the G-invariant part of $\mathscr{H}_k$. See also [**?**].

Theorem 1.7. *Let* $\{\xi_1, \ldots, \xi_N\}$ *be any orthonormal basis of* $\mathbb{C}^N$*. On* $\mathscr{F}_k(\mathbb{C}^N)$*, the corresponding operator to* *the Hamiltonian $\mathscr{H}_k$ is given by*

$$\widetilde{\mathscr{H}_k} = (\gamma_k + N/2) + \sum_{i=1}^{N} \xi_i \partial_{\xi_i},$$

where $\gamma_k = \sum_{\alpha \in \mathscr{R}^+} k_\alpha$.

We now describe the structure of a representation of the universal covering group $\widetilde{SL(2,\mathbb{R})}$ of $SL(2,\mathbb{R})$ on $\mathscr{P}(\mathbb{C}^N)$. This representation, together with the left regular action of the Coxeter group G, allows to obtain the branching decomposition of the Fock space $\mathscr{F}_k(\mathbb{C}^N)$ under the action of $G \times \widetilde{SL(2,\mathbb{R})}$.

Choose $z_1, z_2, \dots, z_N$ as the usual system of coordinates on $\mathbb{C}^N$. Let

$$\mathbb{E} = \frac{1}{2}\sum_{i=1}^{N} z_i^2, \quad \mathbb{F} = -\frac{1}{2}\Delta_k,$$

$$\mathbb{H} = N/2 + \gamma_k + \sum_{i=1}^{N} z_i \partial_{z_i}.$$

In the notation above, the operator $\mathbb{H} = \widetilde{\mathscr{H}_k}$. Then $\mathbb{E}$ (resp. $\mathbb{F}$) acts on $\mathscr{F}_k(\mathbb{C}^N)$ as a creation (resp. annihilation) operator, and $\mathbb{H}$ acts on $\mathscr{F}_k(\mathbb{C}^N)$ as a number operator. If $\mathscr{P}(\mathbb{C}^N) = \bigoplus_{m=0}^{\infty} \mathscr{P}_m(\mathbb{C}^N)$ is the natural grading on $\mathscr{P}(\mathbb{C}^N)$, it is clear that $\mathbb{E}$ raises $\mathscr{P}_m(\mathbb{C}^N)$ to $\mathscr{P}_{m+2}(\mathbb{C}^N)$, $\mathbb{F}$ lowers $\mathscr{P}_m(\mathbb{C}^N)$ to $\mathscr{P}_{m-2}(\mathbb{C}^N)$, and $\mathbb{H}$ multiplies (elementwise) $\mathscr{P}_m(\mathbb{C}^N)$ by the number $(N/2 + \gamma_k + m)$. Heckman showed the following commutation relations

$$[\mathbb{E}, \mathbb{F}] = \mathbb{H}, \quad [\mathbb{E}, \mathbb{H}] = -2\mathbb{E}, \quad [\mathbb{F}, \mathbb{H}] = 2\mathbb{F}. \tag{1.2}$$

These are the commutation relations of a standard basis of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. This gives rise to an infinitesimally unitary representation ϖ_k of $\mathfrak{sl}(2, \mathbb{R})$. The unitarity of ϖ_k follows from the fact that $\mathbb{E}^* = -\mathbb{F}$ and $\mathbb{H} = \mathbb{H}^*$ Notice also that $\mathbb{H}$ has discrete spectrum bounded below.

Denote by $\mathscr{W}_{m+N/2+\gamma_k}$ the unitary representation of $SL(2,\mathbb{R})$ with lowest weight $m+N/2+\gamma_k$.

For $m \in \mathbb{N}$, set $\mathscr{H}_m(k)$ to be the space of harmonic homogeneous polynomials of degree m, i.e. all functions $p \in \mathscr{P}_m(\mathbb{C}^N)$ such that $\Delta_k p = 0$.

Now one of the key features in this formalism is the following branching decomposition.

Theorem 1.8. *The space $\mathscr{P}_m(\mathbb{C}^N)$ of homogeneous polynomials of degree m has a unique decomposition of the form*

$$\mathscr{P}_m(\mathbb{C}^N) = \sum_{\mu=0}^{[m/2]}{}^{\oplus} \langle z, z\rangle^{\mu} \mathscr{H}_{m-2\mu}(k),$$

where $\mathscr{H}_{m-2\mu}(k)$ denotes the space of harmonic homogeneous polynomials of degree $m-2\mu$. Moreover, each homogeneous polynomial $p \in \mathscr{P}_m(\mathbb{C}^N)$ can be written *in a unique way as*

$$p(z) = \sum_{\mu=0}^{[m/2]} \frac{\Gamma(N/2+m-\mu+\gamma_k-1)}{4^\mu \Gamma(\mu+1)\Gamma(N/2+m+\gamma_k-1)}$$

$$\langle z, z\rangle^\mu h_{m-2\mu}(z),$$

where $h_{m-2\mu} \in \mathscr{H}_{m-2\mu}(k)$ *and is given explicitly by*

$$h_{m-2\mu}(z) =$$

$$\sum_{\nu=0}^{[m/2]-\mu} \frac{(-1)^\nu \Gamma(N/2+m-2\mu-\nu-1+\gamma_k)}{4^\nu \Gamma(\nu+1)\Gamma(N/2+m-2\mu+\gamma_k-1)}$$

$$\langle z, z\rangle^\nu \Delta_k^{\mu+\nu} p(z).$$

For $g \in G$, denote by $\pi(g)$ the left regular action of G on $\mathscr{F}_k(\mathbb{C}^N)$

$$\pi(g)f(z) = f(g^{-1}z).$$

The actions of G and $\mathfrak{sl}(2,\mathbb{R})$ on $\mathscr{F}_k(\mathbb{C}^N)$ commute.

We now summarize the consequences of all the above computations and discussions. .

Theorem 1.9. *As a $G\times \widetilde{SL(2,\mathbb{R})}$-module, the Fock space admits the following multiplicity-free decomposition*

$$\mathscr{F}_k(\mathbb{C}^N) = \bigoplus_{m=0}^{\infty} \mathscr{H}_m(k) \otimes \mathscr{W}_{m+N/2+\gamma_k}, \tag{1.3}$$

where $\mathscr{W}_{m+N/2+\gamma_k}$ is the $\widetilde{SL(2,\mathbb{R})}$-representation with lowest weight $m+N/2+\gamma_k$. We also have the following separation of variables decomposition

$$\mathscr{P}(\mathbb{C}^N) = \sum_{m=0}^{\infty}{}^{\oplus} \sum_{\mu=0}^{[m/2]}{}^{\oplus} \langle z, z\rangle^{\mu} \mathscr{H}_{m-2\mu}(k).$$

2. Some applications of the $\mathfrak{sl}(2)$-triple

In this section we will give several applications of the $\mathfrak{sl}(2,\mathbb{R})$-representation discussed in the previous section. In the first and the second applications we adapt the method of R. Howe in the theory of ordinary derivatives, i.e. when $k \equiv 0$.

I. ***A Bochner formula for the Dunkl transform.*** Denote by $\mathscr{S}(\mathbb{R}^N)$ the Schwartz space of rapidly decreasing functions equipped with the usual Fréchet space topology. From Theorem 2.11 it follows, by standard arguments, that

$$\mathscr{S}(\mathbb{R}^N) = \sum_{m=0}^{\infty}{}^{\oplus} \mathscr{H}_{m,\mathbb{R}}(k) \cdot \mathscr{I}(\mathbb{R}^N), \tag{2.1}$$

where $\mathscr{I}(\mathbb{R}^N)$ denotes the space of $O(N)$-invariant Schwartz functions on $\mathbb{R}^N$, and $\mathscr{H}_{m,\mathbb{R}}(k)$ is the space of harmonic homogeneous polynomials in $\mathscr{P}(\mathbb{R}^N)$ of degree m. By abuse of notation we will write $\mathscr{H}_m(k)$ for $\mathscr{H}_{m,\mathbb{R}}(k)$. In the light of (3.1) we may consider the map

$$\zeta^N_{m,k} : \mathscr{H}_m(k) \otimes \mathscr{S}(\mathbb{R}^+) \to \mathscr{S}(\mathbb{R}^N),$$

defined by

$$\zeta^N_{m,k}(h_m \otimes \psi)(x) := h_m(x)\psi(\|x\|^2),$$

$$h_m \in \mathscr{H}_m(k), \quad \psi \in \mathscr{S}(\mathbb{R}^+).$$

By means of the representation ϖ_k we construct a representation $\pi^N_{m,k}$ of $\mathfrak{sl}(2,\mathbb{R})$ on $\mathscr{S}(\mathbb{R}^+)$ as follows

$$\zeta^N_{m,k}\big(h_m \otimes \pi^N_{m,k}(X)\psi\big) = \varpi_k(X)\big(\zeta^N_{m,k}(h_m \otimes \psi)\big),$$

$$X \in \mathfrak{sl}(2,\mathbb{R})$$

for fixed $h_m \in \mathscr{H}_m(k)$. Using (2.4), one may check that

$$\pi^N_{m,k}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}t,$$

$$\pi^N_{m,k}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 2t\frac{d}{dt} + \big(m + \frac{N}{2} + \gamma_k\big),$$

$$\pi^N_{m,k}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = -2\Big\{t\frac{d^2}{dt^2} + \big(m + \frac{N}{2} + \gamma_k\big)\frac{d}{dt}\Big\},$$

where $\gamma_k = \sum_{\alpha \in \mathscr{R}^+} k_\alpha$, and t is the positive variable of $\mathbb{R}^+$. Note that $\pi^N_{m,k}$ does not depend on h_m, and depends only on $m + \frac{N}{2} + \gamma_k$.

The Dunkl transform $\mathscr{D}_k$ can be written as

$$\mathscr{D}_k = e^{i\frac{\pi}{2}(\gamma_k + N/2)} e^{-i\frac{\pi}{8}(-\Delta_k + 4\langle x,x\rangle)},$$

whilst $\mathbb{X} := \left\{\frac{1}{4}(-\Delta_k + 4\langle x, x\rangle)\right\}$ is the generator of the Lie algebra $\mathfrak{k} \cong \mathfrak{so}(2)$.

Theorem 2.1. (i) *For* $f(x) = h_m(x)\psi(\|x\|^2)$, *with* $h_m \in \mathscr{H}_m(k)$ *and* $\psi \in \mathscr{S}(\mathbb{R}^+)$, *we have*

$$\mathscr{D}_k(f)(\xi) = h_m(\xi)\mathscr{D}^N_{m,k}(\psi)(\|\xi\|^2),$$

where $\mathscr{D}^N_{m,k}$ *depends only on* $m + \frac{N}{2} + \gamma_k$, *up to a constant, i.e.*

$$e^{-i\frac{\pi}{2}(\gamma_k + \frac{N}{2})}\mathscr{D}^N_{m,k} = e^{-i\frac{\pi}{2}(\gamma_{k'} + \frac{N'}{2})}\mathscr{D}^{N'}_{m',k'}$$

if

$$m + \frac{N}{2} + \gamma_k = m' + \frac{N'}{2} + \gamma_{k'}. \qquad (2.2)$$

(ii) *The transform* $\mathscr{D}^N_{m,k}$ *coincides with the usual Hankel transform. More precisely, for* $\psi \in \mathscr{S}(\mathbb{R}^+)$

$$\mathscr{D}^N_{m,k}(\psi)(r^2) = e^{-i\frac{\pi}{2}m}\mathscr{H}_{m+\frac{N}{2}+\gamma_k-1}(\psi\circ\Upsilon)(r),$$

where $\Upsilon(t) := t^2$ *for* $t \in \mathbb{R}$, *and*

$$\mathscr{H}_\nu f(r) := \int_0^\infty f(s)\frac{J_\nu(rs)}{(rs)^\nu}s^{2\nu+1}ds$$

denotes the Hankel transform, with J_ν is the Bessel function of the first kind. In these circumstances, (i) reads

$$\mathscr{D}_k\big(h_m\psi(\|\cdot\|)\big)(\xi) = e^{-i\frac{\pi}{2}m}h_m(\xi)\mathscr{H}_{m+\gamma_k+\frac{N}{2}-1}(\psi)(\|\xi\|).$$

To prove the statement (ii) above, we start with the case $m = 0$, and then we use (3.2) to deduce the claim for general m. The above theorem generalizes [**?**, Theorem 2.1], as the example below shows.

***Example* 2.2.** (Hecke-type formula) If we choose $\psi(s) = e^{-\frac{s^2}{2}}$, then the following Hecke-type formula for the Dunkl transform holds

$$\mathscr{D}_k\big(e^{-\frac{\|x\|^2}{2}}h_m\big)(\xi) = e^{-i\frac{\pi}{2}m}h_m(\xi)\mathscr{H}_{m+\gamma_k+\frac{N}{2}-1}\big(e^{-\frac{s^2}{2}}\big)(\|\xi\|) = e^{-i\frac{\pi}{2}m}e^{-\frac{\|\xi\|^2}{2}}h_m(\xi).$$

II. *Huygens' principle.* It is well known that propagation of waves is different in the two- and in the three-dimensional spaces. For instance, suppose we make a "noise" located near a point x at time $t = 0$. Thus we can "hear" this noise at a point y at a later time t only if the distance $y-x$ from y to x is less then t. This phenomena holds in all dimensions, but something special happens in the three dimensional space. After the noise is heard, it moves away and leaves no vibration. This is the so-called Huygens principle. In mathematical terms, Huygens' principle can be explained as following. On $\mathbb{R}^{N+1}$, consider the classical wave equation $(\star)$ $\Delta u(x,t) = \partial_{tt} u(x,t)$. For odd $N \geq 3$, the solution of the Cauchy problem for $(\star)$ at every given point x_0 depends only on its Cauchy data in an arbitrary neighborhood on the light cone surface with vertex x_0. Here we will investigate the validity of

Huygens' principle for when Δ is replaced by the Dunkl-Laplacian operator Δ_k.

For a multiplicity function $k \in \mathscr{K}^+$, consider the following Cauchy problem for the wave equation associated with the Dunkl Laplacian operator

$$\Delta_k u_k(x,t) = \partial_{tt} u_k(x,t), \quad (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

$$u_k(x,0) = f(x), \;\; \partial_t u_k(x,0) = g(x). \tag{2.3}$$

A standard argument shows that the solution of the Cauchy problem (3.3) is uniquely given by

$$u_k(x,t) = (P^{(1)}_{k,t} *_k f)(x) + (P^{(2)}_{k,t} *_k g)(x),$$

where, for fixed t,

$$P^{(1)}_{k,t} = \mathscr{D}_k^{-1}\big[\cos(t\|\cdot\|)\big],$$

$$P^{(2)}_{k,t} = \mathscr{D}_k^{-1}\big[\sin(t\|\cdot\|)/\|\cdot\|\big].$$

Here $*_k$ is a generalized translation, and when $k \equiv 0$, $*_0$ coincides with the usual Euclidean convolution. We refer to [**?**] for the definition of $*_k$. In terms of the propagators, Huygens' principle amounts to the fact that $P_{k,t}^{(1)}$ and $P_{k,t}^{(2)}$ are supported on the light cone $\mathscr{C} := \big\{(x,t) \in \mathbb{R}^N \times \mathbb{R} \mid \|x\|^2 - t^2 = 0\big\}$.

Choose $x_1, x_2, \ldots, x_N$ as the usual system of coordinates on $\mathbb{R}^N$. Let

$$\mathbb{E}_{N,1} := \frac{1}{2}(\|x\|^2 - t^2), \quad \mathbb{F}_{N,1} := -\frac{1}{2}(\Delta_k - \partial_{tt}),$$

$$\mathbb{H}_{N,1} := \frac{N+1}{2} + \gamma_k + \sum_{j=1}^{N} x_j \partial_j + t\partial_t.$$

Using (2.3), one may check the following commutation relations

$$[\mathbb{E}_{N,1}, \mathbb{H}_{N,1}] = -2\mathbb{E}_{N,1}, \quad [\mathbb{F}_{N,1}, \mathbb{H}_{N,1}] = 2\mathbb{F}_{N,1},$$
$$[\mathbb{E}_{N,1}, \mathbb{F}_{N,1}] = \mathbb{H}_{N,1}.$$

These are again(!) the commutation relations of a standard basis of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. Equation (3.4) gives rise to a representation Ω_k of $\mathfrak{sl}(2,\mathbb{R})$, which could be described in a similar way to ϖ_k from the previous section.

Define the distributions $P_k^{(\ell)}$ on $\mathbb{R}^{N+1}$ by

$$P_k^{(\ell)}(\psi_1 \otimes \psi_2) := \int_{\mathbb{R}} P_{k,t}^{(\ell)}(\psi_1)\psi_2(t)dt, \qquad \ell = 1, 2,$$

where $\psi_1 \in \mathscr{S}(\mathbb{R}^N)$ and $\psi_2 \in \mathscr{S}(\mathbb{R})$. Thus we may rewrite the solution u_k as

$$u_k = P_k^{(1)} *_{k,x} f + P_k^{(2)} *_{k,x} g,$$

where $*_{k,x}$ is the $*_k$-convolution with respect to x. Since $\mathscr{C}$ is the locus of zeros of $\|x\|^2 - t^2$, then $P_k^{(\ell)}$ ($\ell = 1, 2$) is supported on the light cone $\mathscr{C}$ if and only if

$$(\|x\|^2 - t^2)^m P_k^{(\ell)} = 0, \quad \text{i.e. } \mathbb{E}_{N,1}^m \cdot P_k^{(\ell)} = 0,$$

for some positive integer m. Further, one can prove that

$$(\Delta_k - \partial_{tt})P_k^{(\ell)} = 0, \quad \text{i.e. } \mathbb{F}_{N,1} \cdot P_k^{(\ell)} = 0. \tag{2.4}$$

These two facts yield the following theorem.

Theorem 2.3. *Huygens' principle holds in the sense that $P_k^{(\ell)}$ ($\ell = 1, 2$) is supported on the light cone $\mathscr{C}$ if and only if*

$P_k^{(\ell)}$ generates a finite-dimensional $\Omega_k^*(\mathfrak{sl}(2,\mathbb{R}))$-module.

In the light of the above theorem, we will look for conditions on N and k which may assure the existence of a finite dimensional representation. As a first step in this direction, when $(N+1)/2+\gamma_k \notin \mathbb{Z}$, Huygens' principle fails. This is due to the fact that the spectrum of the element $\mathbb{H}_{N+1} = (N+1)/2+\gamma_k+\sum_{i=1}^{N+1} x_i\partial_i$ (or its dual) acting on $\mathscr{S}(\mathbb{R}^{N+1})$ (or $\mathscr{S}'(\mathbb{R}^{N+1})$) is $(N+1)/2+\gamma_k+\mathbb{Z}$, whilst the spectrum of $\mathbb{H}_{N+1}$ (or its dual) in finite dimensional modules is contained in $\mathbb{Z}$.

This leaves the possibility that Huygens' principle may hold when $(N+1)/2+\gamma_k \in \mathbb{Z}$, which turns out to be true. To see this, let us first define the dilation operator S_λ on $\mathscr{S}(\mathbb{R}^{N+1})$ by $S_\lambda\psi(x,t) = \psi(\lambda x, \lambda t)$, for $\lambda > 0$. By duality, S_λ acts on distributions in the standard way. Thus, we prove that

$$S_\lambda P_k^{(\ell)} = \lambda^\ell P_k^{(\ell)}.$$

Second, we introduce what we call the Dunkl-Fourier transform

$$\mathscr{D}_k\mathscr{F}\psi(x,t) := (2\pi)^{-1/2}c_k^{-1}$$

$$\int_{\mathbb{R}^{N+1}} \psi(y,s)E_k(-ix,y)e^{ist}\omega_k(y)dyds,$$

for $\psi \in \mathscr{S}(\mathbb{R}^{N+1})$. Here we used notation from the previous section. The transform $\mathscr{D}_k\mathscr{F}$ acts on distributions in the standard way. In particular, we show that

$$(\|x\|^2 - t^2)\mathscr{D}_k\mathscr{F}(P_k^{(\ell)}) = 0, \quad \text{i.e.}$$

$$\mathbb{E}_{N,1} \cdot \mathscr{D}_k\mathscr{F}(P_k^{(\ell)}) = 0,$$

and

$$S_\lambda\big(\mathscr{D}_k\mathscr{F}(P_k^{(\ell)})\big) = \lambda^{2\gamma_k+N+1-\ell}\mathscr{D}_k\mathscr{F}(P_k^{(\ell)}).$$

As a consequence of the above discussions, and in the light of (3.5), the following theorem holds.

Theorem 2.4. *Under the assumption*

$$\frac{N+1}{2} + \gamma_k - \ell \in \mathbb{N}, \qquad (2.5)$$

the tempered distribution $P_k^{(\ell)}$ *generates an* $\mathfrak{sl}(2,\mathbb{R})$*-module of dimension*

$$d(k,\ell) = \frac{N+3}{2} + \gamma_k - \ell,$$

with highest weight vector $\mathscr{D}_k \mathscr{F}(P_k^{(\ell)})$ *of highest weight* $\left(\frac{N+1}{2} + \gamma_k - \ell\right)$. *Further, for each* ℓ *there exists a constant* α_ℓ *such that*

$$P_k^{(\ell)} = \alpha_\ell \, \mathbb{F}_{N,1}^{d(k,\ell)-1} \cdot \mathscr{D}_k \mathscr{F}(P_k^{(\ell)}).$$

Theorem 2.5. *Let* u_k *be the solution of the Cauchy problem (3.3) and suppose that the Cauchy data* (f,g) *are supported inside the closed ball of radius* $R > 0$ *about the origin. The support of* u_k *is contained in the conical shell*

$$\mathscr{C} = \left\{(x,t) \in \mathbb{R}^N \times \mathbb{R} \mid |t| - R \le \|x\| \le |t| + R\right\}$$

if and only if

$$(N-3)/2+\gamma_k \in \mathbb{N}.$$

The shell $\mathscr{C}$ *is the union*

$$\bigcup_{\|y\|\le R} \mathscr{C}_y \tag{2.6}$$

where $\mathscr{C}_y$ *is the light cone*

$$\mathscr{C}_y = \big\{(x,t) \mid \|x-y\| = |t|\big\}.$$