Two index theorems on forms

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Introduction

The Atiyah-Singer index theorem proved by M. Atiyah and I.Singer in 1963 states that for an elliptic differential operator on a compact manifold, the analytical index related to the dimension of the space of solutions is equal to the topological index defined in terms of some topological data. It includes many important theorems such as the Riemann-Roch theorem and has applications in theoretical physics.
We describe two simple instances of the index theorem:

- The Euler characteristic of a compact oriented manifold is realised as the analytic index of $D = d + d^*$ considered as a map from even to odd forms, with d the exterior differential on forms.
- The Hirzebruch signature theorem relates the signature of a compact smooth manifold of dimension $4k$ corresponding to the analytic index of a certain Dirac operator on forms, with the L-genus of the manifold.

Since these lectures are intended for non experts, we do not aim at proving these theorems but rather at providing the necessary geometric tools to understand their statement.

1 Vector bundles and tensor fields

1.1 Definition and first properties

Useful references are [Hus], [KN], [Na].

Definition 1 *A (real or complex) vector bundle of class C^k is a fibre bundle of class C^k with typical fibre a (real or complex) vector space V.*
In other words, it is a triple (E, B, π) often denoted by $\pi : E \to B$ where

- *E and B are differentiable manifolds of class C^k, called total space and base space respectively,*

- *$\pi : E \to B$ is a map of class C^k, called canonical projection, such that there is a set of local charts $(U_i, \phi_i)_{i\in I}$ covering B and C^k diffeomorphisms*

  $$\tau_i : \pi^{-1}(U_i) \to \phi_i(U_i) \times V$$

  satisfying the following requirements:

  i) the fibre $F_b = \pi^{-1}(b)$ is a vector space and

  ii) $\tau_i(b) := \tau_{i|_{F_b}}$ is an isomorphism from F_b to V.

A triple (U_i, ϕ_i, τ_i) is called a local trivialisation of the bundle.
Two local triviallisations (U_i, ϕ_i, τ_i) and (U_j, ϕ_j, τ_j) give rise to maps $\tau_{ij} := \tau_i \circ \tau_j^{-1}$ called transition maps of the form:

$$\begin{array}{rcl} \tau_{ij} : (U_i \cap U_j) \times V & \to & (\phi_i(U_i) \cap \phi_j(U_j)) \times V \\ (b, v) & \mapsto & (b, \tau_{ij}(b)(v)) \end{array}$$

where the $\tau_{ij}(b)$ lie in $GL(V)$.

When $V = \mathbb{R}^d$ (resp. $\mathbb{C}^d$), the vector bundle has rank d. If $d = 1$ it is called a line bundle.
The family $\{\tau_{ij}\}$ is called cocycle associated to the triviallisation $\{U_i, \tau_i, i \in I\}$, and

$$\tau_{ij} \circ \tau_{jk} \circ \tau_{ki} = 1$$

a cocycle relation.

A C^k section of a fibre bundle $\pi : F \to B$ is a map $s : B \to F$ of class C^k such that $\pi \circ s = Id_B$. It is smooth when it is of class C^k for all $k \in \mathbb{N}$.
A real finite rank vector bundle is orientable provided it has a triviallisation with transition maps $\tau_{ij}(b)$ with positive determinant. A manifold is orientable whenever its tangent bundle is orientable.

Remark 1 *In the following we mainly consider smooth manifolds and smooth bundles as well as smooth sections.*

Remark 2 *From a covering of a manifold B together with a set of transition maps satisfying these relations one can reconstruct the fibre bundle on B.*

Example 1 *Given a manifold M of class C^{k+1} (resp. of class C^∞) modelled on a space V, the tangent bundle TM is a C^k (resp. C^∞)-vector bundle with fibres modelled on that same space V; given a local triviallisation (U_i, ϕ_i) on M, a local triviallisation (U_i, ϕ_i, τ_i) on TM is given by $(U_i, \phi_i, D\phi_i)$ and $D\phi_i \circ D\phi_j^{-1}$ is of class C^{k-1}.*

Definition 2 *A morphism of C^k fibre vector bundles $\pi : E \to B$ and $\pi' : E' \to B'$ is a couple (f_0, f) of C^k morphisms $f_0 : B \to B'$ and $f : E \to E'$ such that $\pi' \circ f = f_0 \circ \pi$ and the induced map on the fibres $f_b : \pi^{-1}(b) \to (\pi')^{-1}(b)$ is a linear morphism of the fibres.*

In what follows we shall often take $B = B'$.

Two vector bundles are *isomorphic* if there is a diffeomorphism from one to the other.
A *trivial vector bundle* is vector bundle isomorphic to the bundle $\pi : E = B \times V \to V$.

Definition 3 *Let $\phi : B' \to B$ be a C^k morphism of Banach manifolds, and let $E \to B$ be a C^k vector bundle over B. The pull-back $\phi^* E$ of E by ϕ is a fibre bundle $\phi^*\pi : \phi^* E \to B'$ with total space:*

$$\phi^* E := \{(b', v(\phi(b'))) \in B' \times E_{\phi(b'))}\}$$

where V is the model space of E and with projection $\phi^*\pi(b) = \pi(\phi(b))$.

The set of C^k-sections (resp. C^∞-sections) of a vector bundle E forms a vector space denoted by $C^k(E)$ (resp. $C^\infty(E)$).

Vector fields on a smooth manifold M are smooth sections of the tangent vector bundle. When ϕ is a diffeomorphism, the pull-back $\phi^*\xi$ of a vector field is a section of the pull-back $\phi^* TM$ of the tangent bundle to M and $\phi^*\xi(\phi(m)) = D\phi(\xi(m))$.

1.2 Tensor, dual and morphism bundles

Definition 4 *Let $\pi_1 : E_1 \to B$ and $\pi_2 : E_2 \to B$ be two vector bundles of class C^k with fibres modelled on V_1 and V_2 respectively. The tensor product $\pi_1 \otimes \pi_2 : E_1 \otimes E_2 \to B$ is a vector bundle of class C^k modelled on $V_1 \otimes V_2$ with fibre $\pi_1^{-1}(b) \otimes \pi_2^{-1}(b)$ above $b \in B$ and the local triviallisations of which are built from the tensor product of local triviallisations (U_i, ϕ_i, τ_i^1), (U_i, ϕ_i, τ_i^2) and $(U_i, \phi_i, \tau_i^1 \otimes \tau_i^2)$.*

Transition functions are given by tensor products $\tau_{ij}^1 \otimes \tau_{ij}^2$ where $\tau_{ij}^k, k = 1, 2$ are transition maps for the bundles $E_k, k = 1, 2$.
Whenever E_1 and E_2 have ranks d_1 and d_2, their tensor product has rank $d_1 d_2$.

Given a topological vector space V, the dual space V^* is the space of continuous linear forms on V.

Definition 5 *Let $\pi : E \to B$ be a C^k vector bundle with fibres modelled on a Banach space V. The dual bundle $\pi^* : E^* \to B$ is a vector bundle of class C^k modelled on V^* with fibre $\left(\pi^{-1}(b)\right)^*$ above $b \in B$ and local triviallisations $(U_i, \phi_i, \left(\tau_i^{-1}\right)^*)$ induced by some local triviallisation (U_i, ϕ_i, τ_i) of E. The transition maps are given by $\left(\tau_{ij}^{-1}\right)^*$, where the τ_{ij} are transition maps for E.*

Combining duals and tensor products yields different types of bundles which are useful for geometric purposes. The homomorphism bundle is one of them:

Definition 6 *Given two vector bundles $E \to B$ and $F \to B$, we can build the bundle $Hom(E, F) := E^* \otimes F$ of linear morphisms from E to F. When $E = F$ we denote it by $End(E)$.*

Also we shall use the notion of symmetrized and antisymmetrized tensor products of vector bundles:

Definition 7 *Given vector bundles $E_1, \cdots, E_k$ based on some manifold B, we can build symmetric sections of their tensor product from sections $\sigma_1, \cdots, \sigma_k$ of $E_1, \cdots, E_k$:*

$$\sigma_1 \otimes_s \sigma_2 \otimes_s \cdots \otimes_s \sigma_k := \frac{1}{k!} \sum_{\alpha \in \Sigma_k} \sigma_{\alpha(1)} \otimes \sigma_{\alpha(2)} \otimes \cdots \otimes \sigma_{\alpha(k)},$$

and similarly antisymmetric sections:

$$\sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_k := \frac{1}{k!} \sum_{\alpha \in \Sigma_k} (-1)^{sign(\alpha)} \sigma_{\alpha(1)} \otimes \sigma_{\alpha(2)} \otimes \cdots \otimes \sigma_{\alpha(k)}$$

where $sign(\alpha)$ is the signature of the permutation.

Another useful class of bundles is that of tensor bundles on a manifold:

Definition 8 *Given an n-dimensional manifold X of class C^k then:*

- *The dual bundle T^*X to the tangent bundle TX is a vector bundle called the cotangent bundle. It is a vector bundle of class C^{k-1} over $B = X$ of rank n. Its sections are called cotangent vector fields.*
- *The tensor bundle $TX^q := \otimes^q TX$, $q \in \mathbb{N}^*$ is a vector bundle of class C^{k-1} over $B = X$ of rank qn. Its sections are called contravariant q-tensor fields.*
- *The tensor bundle $(TX^*)^p := \otimes^p TX$, $p \in \mathbb{N}^*$ is a vector bundle of class C^{k-1} over $B = X$ of rank qn. Its sections are called covariant p-tensor fields.*
- *A (p, q) tensor field is a section of the bundle $(\otimes^q TX) \otimes (\otimes^p T^*X)$.*
  A p-form on M is an antisymmetric p-covariant tensor on M.

Remark 3 *In finite dimensions, one often writes a (p, q) tensor T in local coordinates as $T_{i_1 \cdots i_p}^{j_1 \cdots j_q}$.*

Pull-backs can be extended to covariant tensor fields.
Given a morphism $\phi : X \to Y$ between two C^k manifolds X and Y, the pull-back by ϕ of a covariant p-tensor field T on Y is given by:

$$(\phi^* T)_x(U_1, \cdots, U_p) := T_{\phi(x)}(D_x\phi(U_1), \cdots, D_x\phi(U_p)) \quad \forall U_1, \cdots, U_p \in T_x X.$$

In particular, the pull-back of a p-form is also a p-form. It is easy to check that

$$\phi^*(T_1 \otimes T_2) = \phi^* T_1 \otimes \phi^* T_2$$

and that given two morphisms ϕ, ψ we have:

$$(\phi \otimes \psi)^* = \psi^* \otimes \phi^*.$$

If ϕ is a diffeomorphism, the pull-back can be extended to contravariant vector fields:

$$\phi^*(\xi_1 \otimes \cdots \xi_q) := (\phi^{-1})_* \xi_1 \otimes \cdots \otimes (\phi^{-1})_* \xi_q.$$

1.3 Riemannian metric

Useful references are [GHL], [K], [L].
An important example of covariant tensor fields is given by Riemannian (resp. Hermitian) metrics.

Definition 9 *A Riemannian metric on a smooth real finite rank vector bundle E over a manifold B, is a smooth section g of $E^* \otimes E^*$ such that for any $b \in B$, g_b induces a symmetric positive definite form on each fibre E_b.*

A Hermitian metric on a smooth complex vector bundle over a manifold B, is a smooth section h of $E^* \otimes E^*$ such that for any $b \in B$, h_b induces a Hermitian positive definite form on each fibre E_b.
A Riemannian (Hermitian) metric on a Banach manifold is a Riemannian (Hermitian) metric on the tangent bundle TB.

Given a metric on M of dimension n, a local orthonormal system of coordinates $(x_1, \cdots, x_n)$ around a point x is such that setting $e_i := \frac{\partial}{\partial x_i}$ we have $g_{ij}(x) := g_x(e_i, e_j) = \delta_{ij}$, i.e. the matrix representing g_x in this coordinate system is the identity matrix.
A vector bundle equipped with a Riemannian (resp. Hermitian) metric is called a *Riemannian* (resp. *Hermitian*) vector bundle. A manifold M such that TM is equipped with a Riemannian (resp. Hermitian) metric is called a Riemannian (resp. Hermitian) manifold.
If M is a Riemannian manifold, tensor bundles over M can be equipped with a metric structure induced from that of M.
Using a smooth partition of unity on the manifold, one can build a metric patching up locally defined positive definite forms.
Another way of equipping a manifold with a metric is by pull-back. Given a smooth map $\phi : N \to M$ between two manifolds and a Riemannian (resp. Hermitian) metric g (resp. h) on a vector bundle based on M, the pull-back $\phi^* g$ (resp. $\phi^* h$) yields a Riemannian (resp. Hermitian) metric on the pull-back vector bundle $\phi^* E$ based on N.

1.4 Almost complex structure

Definition 10 *An almost complex structure on an oriented vector bundle $\pi : E \to B$ is a smooth section J of $E^* \otimes E$ which induces a morphism, also denoted by J, preserving orientation and such that $J^2 = -Id$. An almost complex structure on an oriented manifold M is one on the tangent bundle TM, i.e. it is a $(1,1)$ tensor J inducing a morphism J on TM which preserves orientation and satisfies $J^2 = -Id$.*

An almost complex structure J on a bundle E gives rise to two subbundles $E^{1,0}$ and $E^{0,1}$ with fibre $Ker(J(b) - I) := \{u_b, J(u_b) = u_b\}$, resp. $Ker(J(b) - I) := \{v_b, J(v_b) = -v_b\}$ above $b \in B$.

If a manifold M is equipped with an almost complex structure such that $TM^{1,0}$ is stable under brackets of vector fields, then M can be equipped with an atlas (U_i, ϕ_i) with transition maps given by holomorphic maps and one calls such an atlas a complex structure on M. Then we call M a complex manifold. Conversely, such a manifold inherits an almost complex structure from the local charts.

If M and N are two complex manifolds, a map $f : M \to N$ is called holomorphic if it *is holomorphic in any local chart, this requirement being independent of the choice of local chart since the transition maps are holomorphic.*

Example 2 *Letting* $\{z_k := x_k + iy_k, k = 1, \cdots, n\}$ *be a system of local coordinates on the complex manifold* M*, we can set*

$$J\left(\frac{\partial}{\partial x_k}\right) = \frac{\partial}{\partial y_k}, \quad J\left(\frac{\partial}{\partial y_k}\right) = -\frac{\partial}{\partial x_k}.$$

This defines a $(1,1)$ *tensor on* M *independently of the choice of local coordinates. Indeed, given another system of local coordinates* $\{z'_k := x'_k + iy'_k, k = 1, \cdots, n\}$*, the Cauchy-Riemann equations*

$$\frac{\partial x_i}{\partial x'_j} = \frac{\partial y_i}{\partial y'_j}, \quad \frac{\partial x_i}{\partial y'_j} = \frac{\partial y_i}{\partial x'_j}$$

lead to a similar expression

$$J\left(\frac{\partial}{\partial x'_k}\right) = \frac{\partial}{\partial y'_k}, \quad J\left(\frac{\partial}{\partial y'_k}\right) = -\frac{\partial}{\partial x'_k}$$

so that we obtain an almost complex structure J *on* M*.*

A *Riemann surface* is a real 2-dimensional manifold M equipped with a complex structure c. Given a Riemann surface (M, c) and $f : M \to M$ a diffeomorphism, one can produce a new complex structure $f^*c := \{f^{-1}(U_i), \phi_i \circ f\}$ called the *pull-back* of c by f. It is different from the initial one in the sense that the charts are not only different from the initial ones but also incompatible with them. Yet (M, c) and (M, f^*c) are holomorphically equivalent in the sense that $f : (M, c) \to (M, f^*c)$ is a holomorphic map and so is its inverse.

A Riemannian metric g combined with an almost complex structure yields a Hermitian metric:

$$h(\sigma, \rho) := g(\sigma, J\rho).$$

2 Connections on vector bundles

Covariant derivatives extend the exterior differentiation to sections of vector bundles.

Definition 11 *Given a vector bundle* $\pi : E \to B$ *based on a manifold* B*, a covariant derivative (also abusively called connection) on* E *is a differential operator:*

$$\nabla : C^\infty(E) \to C^\infty(T^*B \otimes E)$$

which satisfies the Leibniz rule:

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma.$$

It extends in a unique way to the space $\Omega(B, E)$ *of* E*-valued forms on* B*:*

$$\nabla(\alpha \wedge \theta) := d\alpha \wedge \theta + (-1)^{|\alpha|}\alpha \wedge \nabla\theta \quad \forall \alpha \in \Omega(B), \beta \in \Omega(B, E).$$

Remark 4 *Notice that for any* $f \in \Omega^0(B)$,$U, V \in C^\infty(TB)$ *we have:*

$$\nabla_{fU}\sigma = f\nabla_U\sigma$$

and

$$\nabla_{U+V}\sigma = \nabla_U\sigma + \nabla_V\sigma \quad \forall\sigma \in C^\infty(E).$$

A covariant derivation ∇ on a vector bundle E induces a *dual connection* ∇^* on the dual bundle E^*, given by the Leibniz rule using the duality product:

$$d\langle\sigma, \rho\rangle = \langle\nabla\sigma, \rho\rangle + \langle\sigma, \nabla^*\rho\rangle, \quad \forall\sigma, \rho \in C^\infty(E),$$

and a connection ∇^{End} on the bundle $\mathrm{End}(E) \simeq E^* \otimes E$ defined by:

$$\nabla^{End} := \nabla^* \otimes 1 + 1 \otimes \nabla.$$

For a trivial vector bundle $E \to B$, any connection is given by an $\mathrm{End}(E)$-valued one form θ via the formula $\nabla = d + \theta$. As a consequence, a connection on a general vector bundle can locally be described by $\nabla = d + \theta_U$ where now θ_U is a $\mathrm{End}(E)$-valued one form on an open subset $U \in B$ over which we have trivialized the bundle. Another consequence is that two connections on E differ by a (globally defined) $\mathrm{End}(E)$-valued one form on B. An easy computation yields that if $\nabla = d+\theta_U$ locally, then $\nabla^* = d-\theta_U$ and $\nabla^{\mathrm{End}} = d+[\theta_U, \cdot]$.

A similar formula to that of the differentiation on ordinary forms holds for a covariant derivative on E-valued forms:

$$\begin{aligned}\nabla\alpha(U_0, \cdots, U_p) &= \sum_{k=0}^{p}(-1)^k\nabla_{U_i}\left(\alpha(U_0, \cdots, \hat{U}_i, \cdots, U_k)\right)\\ &+ \sum_{0\leq k<l\leq p}(-1)^{k+l}\alpha([U_k, U_l], U_0, \cdots, \hat{U}_k, \cdots, \hat{U}_l, \cdots, U_p)\end{aligned}$$

where $\hat{U}_i$ means that we have left out the vector field U_i.

A connection ∇ on a Riemannian (resp. Hermitian) bundle $\pi : E \to B$ based on a manifold B is *Riemannian* provided it is compatible with the Riemannian (resp. Hermitian) metric in the following sense:

$$d\langle\sigma, \rho\rangle_b = \langle\nabla\sigma, \rho\rangle_b + \langle\sigma, \nabla\rho\rangle_b \quad \forall\sigma, \rho \in C^\infty(E) \quad \forall b \in B$$

where $\langle\cdot, \cdot\rangle_b$ is the inner product on the fibre above b.

Given a connection ∇ on a finite n-dimensional manifold M, and given a local system of coordinates $(x_1, \cdots, x_n)$ at a point $x \in M$, we define the *Christoffel symbols:*

$$\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j} = \sum_{k=1}^{n}\Gamma_{ij}^k\frac{\partial}{\partial x_k}.$$

Definition 12 *The torsion of a connection on the tangent bundle* TM *to a manifold* M *is given by:*

$$T(U, V) := \nabla_U V - \nabla_V U - [U, V] \quad \forall U, V \in TM.$$

In a system of local coordinates $(x_1, \cdots, x_n)$ *around a point* x *of a finite* n*-dimensional manifold* M*, setting* $e_i = \frac{\partial}{\partial x_i}$ *we have* $T(e_i, e_j) = \nabla_{e_i}e_j - \nabla_{e_j}e_i$ *so that if the torsion vanishes then* $\nabla_{e_i}e_j = \nabla_{e_j}e_i$*, i.e.* $\Gamma_{ij}^k = \Gamma_{ji}^k$.

Proposition 1 *There is a unique connection on a Riemannian manifold which has vanishing torsion and is compatible with the (strong) Riemannian metric; it is called the Levi-Civita connection.*

Idea of proof: Writing $d\langle U,V\rangle(W) = \langle \nabla_W U, V\rangle + \langle U, \nabla_W V\rangle$ as well as circular combinations of this expression and using the fact that the torsion vanishes yields the expression of $\langle \nabla_W U, V\rangle$ in terms of differentials of the inner product $\langle U,V\rangle$, $\langle V,W\rangle$ and $\langle U,W\rangle$. The existence and uniqueness of $\nabla_W U$ then follows from Riesz's theorem. $\square$

Definition 13 *The curvature of a covariant derivation is given by the* End(E)*-valued two form* $\Omega = \nabla^2 \in \Omega^2(B, \mathrm{End}(E))$*:*

$$\Omega(U,V) := [\nabla_U, \nabla_V] - \nabla_{[U,V]} \quad \forall U,V \in C^\infty(B, TB).$$

An easy computation shows that the curvature is a local operator, meaning by this that $\Omega(U,V)f = f\Omega(U,V)$, although one could expect apriori from the above formula that f might get differentiated.

It is clear from the definition of the curvature that the *Bianchi identity*

$$[\nabla, \Omega] = 0$$

holds.
The *Ricci tensor* of a connection ∇ on a Riemannian manifold M is defined by

$$R(X,Y,W,Z) := \langle \Omega(X,Y)W, Z\rangle$$

where X,Y,W,Z are vector fields on M and $\langle\cdot,\cdot\rangle$ the inner product induced by the Riemannian structure. We have:

$$R(X,Y,W,Z) = -R(Y,X,W,Z) = -R(X,Y,Z,W)$$

and

$$R(X,Y,W,Z) = R(W,Z,X,Y).$$

When M is finite dimensional, the *Ricci curvature* is given by the trace of the operator $\Omega(X,\cdot)Y$, i.e. $\mathrm{Ricc}(X,Y) := \mathrm{tr}(\Omega(X,\cdot)Y)$. The *scalar curvature* is the trace of the Ricci curvature $s(x) = \sum_{i=1}^n \mathrm{Ricc}(e_i(x), e_i(x))$, where $(e_i(x))_{i\in\mathbb{N}}$ is any local orthonormal frame of T_xM.

A connection with vanishing curvature is called a *flat* connection. When the Ricci curvature vanishes, the manifold is called *Ricci flat*.

3 Exterior differentiation

A useful reference is [MT].
Exterior differentiation on forms on a given smooth manifold M is defined as follows.

Definition 14 *The derivation* $f \to Df$ *defined on the space of smooth functions on a manifold* M *extends to a unique linear map* $d : \Omega(M) \to \Omega(M)$ *such that*

1. *d sends $\Omega^p(M)$ to $\Omega^{p+1}(M)$,*
2. *$(d \circ d) f = 0 \quad \forall f \in C^\infty(M)$,*
3. *d is a derivation i.e. it satisfies the (graded) Leibniz rule:*

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta \quad \forall \alpha, \beta \in \Omega(M).$$

Remark 5 *The ordinary differentiation on sections of a trivial bundle is flat since $d \circ d = 0$. Writing the connection on a vector bundle in a triviallisation over an open subset U of the base manifold $\nabla = d + \theta_U$, the curvature reads*

$$\Omega = d\theta_U + \theta_U \wedge \theta_U.$$

From these formulas it follows that given a form $\alpha \in \Omega^p(M)$, and given smooth vector fields $U_0, \cdots, U_p$ on M, the exterior differentiation reads:

$$\begin{aligned} d\alpha(U_0, \cdots, U_p) &= \sum_{k=0}^{p} (-1)^k U_i \left(\alpha(U_0, \cdots, \hat{U}_k, \cdots, U_p) \right) \\ &+ \sum_{1 \leq k, l \leq p} (-1)^{k+l} \alpha \left([U_k, U_l], U_0, \cdots, \hat{U}_k, \cdots, \hat{U}_l, \cdots, U_p \right) \end{aligned}$$

where the "hat" above the vector fields means we have deleted them.
A form α is *closed* whenever $d\alpha = 0$, and *exact* whenever there is a form β such that $\alpha = d\beta$. Since $d \circ d = 0$, exact forms are closed and we can form the de Rham cohomology complex:

$$H^p(B) := \mathrm{Ker}(d|_{\Omega^p(B)}) / \mathrm{R}(d|_{\Omega^{p-1}(B)})$$

where $\mathrm{R}(d|_{\Omega^{p-1}(B)})$ denotes the range of the map $d|_{\Omega^{p-1}(B)}$.

3.1 The Hodge star operator

Let M be a finite dimensional oriented Riemannian manifold. There is a natural pointwise inner product on forms induced by the Riemannian structure on M. We first need to define the *Hodge star* operator which yields a duality between p-forms and $(n-p)$-forms.
Recall that a Riemannian structure on a finite n-dimensional oriented manifold M yields a particular form, the *volume form* given in a local system of coordinates $(x_1, \cdots, x_n)$ at a point x by:

$$d\mathrm{vol}(x) = \sqrt{\det g_x}\, dx_1 \wedge \cdots \wedge dx_n,$$

where $\det g_x$ is the positive determinant of the matrices representing the metric locally at point x. The Hodge star operator is a bundle morphism defined pointwise as the linear operator $* : \Lambda^p T_x^* M \to \Lambda^{n-p} T_x^* M$ on a positively orthonormal local oriented basis $\{e_1^*, \cdots, e_n^*\}$ of $T_x^* M$ by:

$$e_{i_1}^* \wedge \cdots \wedge e_{i_p}^* \wedge \star(e_{i_1}^* \wedge \cdots \wedge e_{i_p}^*) = d\mathrm{vol}(x) \quad \forall i_1 < \cdots < i_p.$$

This definition is independent of the choice of the oriented orthonormal basis and one can check that $\star^2 = (-1)^{p(n-p)}$ on $\Lambda^p T_x^* M$.

Equivalently, the Hodge star operator $\star$ satisfies the following requirement:

$$\langle\alpha,\beta\rangle_x d\mathrm{vol}(x) = \alpha(x)\wedge\star\beta(x) \quad \forall\alpha,\beta\in\Lambda^p T_x^* M,$$

from which the symmetry property $\alpha\wedge\star\beta = \beta\wedge\star\alpha$ immediately follows. Notice that if f is a smooth function then $\star f = fe_1^*\wedge\cdots\wedge e_n^*$ where $e_i^*, i = 1,\cdots,n$ is an orthonormal basis of T^*M.

When M is moreover compact, it yields an inner product on forms seen as sections:

$$\langle\alpha,\beta\rangle_g = \int_M \langle\alpha,\beta\rangle_x d\mathrm{vol}(x) = \int_M \alpha(x)\wedge\star\beta(x). \tag{1}$$

In local coordinates, writing $\alpha = \alpha_I dxi_I$, $\beta = \beta_J dx_J$ where $I = \{i_1,\cdots,i_p\}$, $J = \{j_1,\cdots,j_p\}$ are two multiindices of length p, we have

$$\langle\alpha,\beta\rangle_g = g^{i_1,j_1}\cdots g^{i_p j_p}\alpha_{i_1,\cdots,i_p}\beta_{j_1,\cdots,j_p}.$$

Here g^{ij} is the $(ij)-th$ coefficient of the inverse matrix of g_{ij} and following Einstein's conventions, one sums over repeated indices.

Using the inner product on forms, one can define the (formal) adjoint $d^* : \Omega^{p+1}(M)\to\Omega^p(M)$ of the exterior differentiation $d : \Omega^p(M)\to\Omega^{p+1}(M)$ setting:

$$\langle d\alpha,\beta\rangle_g := \langle\alpha, d^*\beta\rangle \quad \forall\alpha\in\Omega^p(M), \beta\in\Omega^{p+1}(M).$$

3.2 Musical isomorphisms

The Riemannian metric on M also yields a duality correspondence often referred to as "musical" isomorphisms. Given $v\in T_xM$, $v^\flat\in T_x^*M$ is uniquely defined by

$$v^\flat(w) = \langle v,w\rangle_x.$$

In local coordinates, this corresponds to lowering indices:

$$\begin{array}{rcl} T_xM & \to & T_x^*M \\ v = (v^1,\cdots,v^n) & \mapsto & v^\flat = (\sum_{j=1}^n g_{1j}v^j,\cdots,\sum_{i=1}^n g_{nj}v^j) \end{array}$$

where (g_{ij}) is the (locally defined) metric matrix. In a similar way, given $\alpha\in T_x^*M$, $\alpha^\sharp\in T_xM$ is uniquely defined by

$$\alpha(w) = \langle\alpha^\sharp, w\rangle_x$$

using the Riesz theorem. In local coordinates, this corresponds to raising indices:

$$\begin{array}{rcl} T_x^*M & \to & T_xM \\ \alpha = (\alpha_1,\cdots,\alpha_n) & \mapsto & \alpha^\sharp = (\sum_{j=1}^n g^{1j}\alpha_j,\cdots,\sum_{j=1}^n g^{nj}\alpha_j) \end{array}$$

where (g^{ij}) is the (locally defined) inverse metric matrix.

3.3 The interior and exterior products on forms

The *interior product* is a bundle morphism:

$$\begin{array}{rcl} T_xM \times \Lambda^p T_x^* M & \to & \Lambda^{p-1} T_x^* M \\ (v, \alpha) & \mapsto & i_v(\alpha) := \alpha(v, \cdots). \end{array}$$

It satisfies:

$$i(v)(\alpha \wedge \beta) = i(v)(\alpha) \wedge \beta + (-1)^p \alpha \wedge i(v)\beta, \forall v \in T_xM, \alpha \in \Lambda^p T_x^* M.$$

In a local coordinate chart $(x_1, \cdots, x_n)$ on M we have:

$$i(v)(dx_1 \wedge \cdots \wedge dx_p) = \sum_{i=1}^{p} (-1)^{i+1} dx_i(v)\, dx_1 \wedge \cdots \wedge d\hat{x}_i \wedge \cdots dx_p, \quad \forall v \in T_xM.$$

Given a connection ∇^E on a bundle E based on M, set $\nabla^E_V := i(V) \circ \nabla$. Combining the graded Leibniz rule for ∇^E extended to E-valued forms with the graded Leibniz rule for $i(V)$ we get for any $\alpha, \beta \in \Omega(M, E)$:

$$\nabla^E_V(\alpha \wedge \beta) = \nabla^E_V \alpha \wedge \beta + \alpha \wedge \nabla^E_V \beta.$$

Let M be a Riemannian manifold, the *exterior product* is a bundle morphism:

$$\begin{array}{rcl} TM \times \Lambda^p T^* M & \to & \Lambda^{p+1} T^* M \\ (v, \alpha) & \mapsto & \epsilon(v)(\alpha) := v^\flat \wedge \alpha. \end{array}$$

It is easy to check that $\epsilon(v)\epsilon(w) + \epsilon(w)\epsilon(v) = 0$ for all $v, w \in T_xM$, and that $i(v)i(w) + i(w)i(v) = 0$ for all $v, w \in T_xM$.

4 Invariant polynomials

4.1 First definitions

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. G acts on itself by conjugation:

$$\begin{array}{rcl} G \times G & \to & G \\ (g, h) & \mapsto & g^{-1}\, h\, g \end{array}$$

and induces the adjoint action on $\mathfrak{g}$:

$$\begin{array}{rcl} Ad : G \times \mathfrak{g} & \to & \mathfrak{g} \\ (g, u) & \mapsto & Ad_g(u) := \frac{d}{dt}_{|_{t=0}} C_g(h_t) \end{array}$$

where $h_t = e^{tu}$ is a local one parameter group generated by $u \in \mathfrak{g}$. This in turn induces a tangent action:

$$\begin{aligned} ad : \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g} \\ (v, u) &\mapsto ad_v(u) := \frac{d}{dt}_{|t=0} Ad_{g_t}(u) \end{aligned}$$

where $g_t = e^{tv}$ is a local one parameter group generated by $v \in \mathfrak{g}$.

In the case where $G = GL_n(\mathbb{C})$ is the group on invertible $n \times n$ matrices with complex coefficients and $\mathfrak{g} = gl_n(\mathbb{R})$ the algebra of all $n \times n$ matrices with complex coefficients we have

$$ad_v(u) = [u, v], \quad \forall u, v \in \mathfrak{g}.$$

Definition 15 *An G- invariant polynomial on $\mathfrak{g}$ is a smooth map $P : \mathfrak{g} \to K$ where K is the base field of $\mathfrak{g}$ such that*

$$P(Ad_g a) = P(a) \quad \forall a \in \mathfrak{g}.$$

Since the map $t \to P(ta)$ is smooth, it has a Taylor expansion at all orders:

$$P(ta) = \sum_{k=0}^{K} \left(\frac{\partial^k}{\partial t} P(ta) \right)_{t=0} \frac{t^k}{k!} + o_a(t^K)$$

so that setting $P_k(a) := \left(\frac{\partial^k}{\partial t} P(ta) \right)_{t=0} \frac{t^k}{k!}$ we have:

$$P(ta) = \sum_{k=0}^{K} P_k(a) t^k + o_a(t^K).$$

Lemma 1

- *If P is invariant, so is the polarized polynomial*

  $$p(a_1, \cdots, a_k) = \frac{1}{k!} \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_k} P(t_1 a_1 + \cdots t_k a_k)_{t_1=0, \cdots, t_k=0}$$

  for any $a_1, \cdots, a_k \in \mathfrak{g}$. In particular, the homogeneous polynomial P_k of degree k is invariant.

- *Given any $a_1, \cdots, a_k \in \mathfrak{g}$,*

  $$p(a_{\sigma(1)}, \cdots, a_{\sigma(k)}) = p(a_1, \cdots, a_k) \quad \forall \sigma \in \Sigma_k.$$

- *The map*

  $$(a_1, \cdots, a_k) \mapsto p(a_1, \cdots, a_k)$$

  is multilinear.

Proof

- Let $g \in G$, then using the invariance of P under the adjoint action of G,

$$\begin{aligned} k!p(Ad_g a_1, \cdots, Ad_g a_k) &= \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_k} P(t_1(Ad_g a_1) + \cdots t_k(Ad_g a_k))_{t_1=0,\cdots,t_k=0} \\ &= \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_k} P\left(Ad_g\left(t_1 a_1 + \cdots t_k a_k\right)\right)_{t_1=0,\cdots,t_k=0} \\ &= \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_k} P\left(t_1 a_1 + \cdots t_k a_k\right)_{t_1=0,\cdots,t_k=0} \\ &= p(a_1, \cdots, a_k). \end{aligned}$$

- Let $j_k = \sigma(k)$ then setting $\rho := \sigma^{-1}$ we get:

$$\begin{aligned} k!p(a_{\sigma(1)}, \cdots, a_{\sigma(k)}) &= \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_k} P(t_1 a_{\sigma(1)} + \cdots t_k a_{\sigma(k)})_{t_1=0,\cdots,t_k=0} \\ &= \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_k} P(t_{\rho(1)} a_1 + \cdots t_{\rho(k)} a_k)_{t_1=0,\cdots,t_k=0} \\ &= \frac{\partial}{\partial t_{\rho(1)}} \cdots \frac{\partial}{\partial t_{\rho(k)}} P(t_{\rho(1)} a_1 + \cdots t_{\rho(k)} a_k)_{t_{\rho(1)}=0,\cdots,t_{\rho(k)}=0} \\ &= k!p(a_1, \cdots, a_k), \end{aligned}$$

  where we have used the fact that the partial derivative $\frac{\partial}{\partial t_i}$ commute.

- We prove linearity for $k = 2$:

$$\begin{aligned} 2p((a_1 + b_1), a_2) &= \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_2} p(a_1 t_1 + b_1 t_1 + b_2 t_2)_{t_1=0,t_2=0} \\ &= \frac{\partial}{\partial t_2} \frac{\partial}{\partial t_1} p(a_1 t_1 + b_1 t_3 + b_2 t_2)_{|t_1=t_2=0} + \frac{\partial}{\partial t_2} \frac{\partial}{\partial t_3} p(a_1 t_1 + b_1 t_3 + b_2 t_2)_{|t_3=t_2=0} \\ &= p(a_1, a_2) + p(b_1, a_2). \end{aligned}$$

□

Corollary 1 *Let $b \in \mathfrak{g}$ and $g_t := e^{tb}$ a one parameter local group of diffeomorphisms generated by b, then*

$$\sum_{i=1}^{k} p(a_1, \cdots, , a_{i-1}, ad_b(a_i), a_{i+1}, \cdots, a_k) = 0.$$

Proof This follows from the invariance of the polarised polynomial which yields:

$$\begin{aligned} 0 &= \frac{d}{dt}_{|t=0} p(Ad_{g_t} a_1, \cdots, Ad_{g_t} a_k) \\ &= \sum_{i=1}^{k} p(a_1, \cdots, ad_b(a_i), \cdots, a_k) \end{aligned}$$

□

4.2 Examples of invariant polynomials

- **The trace on matrices.** Let $G = GL_n(\mathbb{C})$ and $\mathfrak{g} = gl_n(\mathbb{C})$, then $P(a) := \mathrm{tr}(a^k)$ where the product is the product of matrices and the trace the ordinary trace on matrices, defines an invariant polynomial for we have

$$\mathrm{tr}(c^{-1}ac) = \mathrm{tr}(a) \quad \forall c \in GL_n(\mathbb{C}), \forall a \in gl_n(\mathbb{C}).$$

- **The Pfaffian on antisymmetric matrices.** Let $G = SO_n(\mathbb{R})$ and $\mathfrak{g} = so_n(\mathbb{R})$, there is a one to one correspondance:

$$\begin{array}{rcl} \Lambda^2\mathbb{R}^n & \rightarrow & so_n(\mathbb{R}) \\ a_{ij} e_i \wedge e_j & \mapsto & (a_{ij}) \end{array}$$

where $e_i, i = 1, \cdots, n$ is an orthonormal basis for the canonical scalar product on $\mathbb{R}^n$.

Berezin integration on $\Lambda\mathbb{R}^n$ is the linear map defined by:

$$\begin{array}{rcl} T : \Lambda\mathbb{R}^n & \longrightarrow & \mathbb{R} \\ \alpha & \mapsto & e_i^* \wedge \cdots e_n^*(\alpha) \end{array}$$

where $e_i^*, i = 1, \cdots, n$ is the dual basis $e_i^*(e_j) = \delta_{ij}$. T vanishes on $\Lambda^p\mathbb{R}^n$ for any $p < n$ so that for any $v \in \mathbb{R}^n$ and any $\alpha \in \Lambda\mathbb{R}^n$, $T(i(v)\alpha) = 0$ where $i(v)$ is the interior product. The fact that T yields a linear map which vanishes on derivations justifies the terminology "integral" (analogy with Stoke's theorem).

Given a real metric vector bundle E of rank n based on a manifold M, Berezin integration generalizes to a vector bundle morphism:

$$\begin{array}{rcl} T : \Lambda E & \longrightarrow & M \times \mathbb{R} \\ \alpha & \mapsto & e_i^* \wedge \cdots e_n^*(\alpha) \end{array}$$

where $e_i^*, i = 1, \cdots, n$ is now an orthonormal frame of E. T in turn induces a map on sections (denoted by the same symbol) $T : C^\infty(M, \Lambda E) \rightarrow C^\infty(M, \mathbb{R})$ in an obvious way.

Definition 16 *Under the above assumptions on E, the Pfaffian of $A = (a_{ij}) \in C^\infty(M, \Lambda^2 E \simeq so(E))$ is the real valued function on M defined by:*

$$\mathrm{Pf}(A) := T\left(e^{\frac{1}{2}\sum_{i,j=1}^n a_{ij} e_i \wedge e_j}\right) = T\left(e^{\sum_{i<j; i,j=1}^n a_{ij} e_i \wedge e_j}\right).$$

In some cases, the Pfaffian is identified to the top form $\mathrm{Pf}(A) e_1 \wedge \cdots e_n$.

Lemma 2 *Given $A = (a_{ij}) \in C^\infty(M, \Lambda^2 E \simeq so(E))$, if the rank n of E is even, setting $n = 2k$ we have:*

$$\mathrm{Pf}(A) = \frac{(-1)^k}{2^k k!} \sum_{\sigma \in \Sigma_n} \epsilon(\sigma) a_{\sigma(1)\sigma(2)} \cdots a_{\sigma(2k-1)\sigma(2k)}$$

and the Pfaffian vanishes if the rank of E is odd. Here $\epsilon(\sigma)$ denotes the signature of σ.

Proof

$$\begin{aligned}
\mathrm{Pf}(A) &= T\left(e^{\frac{1}{2}\sum_{i,j=1}^n a_{ij} e_i \wedge e_j}\right) \\
&= e_1^* \wedge \cdots e_n^* \left(\frac{1}{2^k k!}\left(\sum_{i,j=1}^n a_{ij} e_i \wedge e_j\right)^k\right) \\
&= e_1^* \wedge \cdots e_n^* \left(\frac{1}{2^k k!} \sum_{\sigma \in \Sigma_n} \epsilon(\sigma) a_{\sigma(1)\sigma(2)} a_{\sigma(3)\sigma(4)} \cdots a_{\sigma(2k-1)\sigma(2k)} e_1 \wedge \cdots \wedge e_{2k}\right) \\
&= (-1)^{\frac{n(n-1)}{2}} \frac{1}{2^k k!} \sum_{\sigma \in \Sigma_n} \epsilon(\sigma) a_{\sigma(1)\sigma(2)} a_{\sigma(3)\sigma(4)} \cdots a_{\sigma(2k-1)\sigma(2k)} \\
&= \frac{(-1)^k}{2^k k!} \sum_{\sigma \in \Sigma_n} \epsilon(\sigma) a_{\sigma(1)\sigma(2)} a_{\sigma(3)\sigma(4)} \cdots a_{\sigma(2k-1)\sigma(2k)}
\end{aligned}$$

using the fact that $e_1^* \wedge \cdots e_n^*(e_1 \wedge \cdots \wedge e_n) = (-1)^{\frac{n(n-1)}{2}} = (-1)^k$.

Proposition 2 *The Pfaffian satisfies the following properties. Let $A \in so_n(\mathbb{R}), B \in GL_n(\mathbb{R})$*

$$\mathrm{Pf}(B^t AB) = \mathrm{Pf}(A)\det(B),$$

in particular,

$$\mathrm{Pf}(B^{-1}AB) = \mathrm{Pf}(A) \quad \text{if} \quad B \in SO_n(\mathbb{R}).$$

Moreover, $\mathrm{Pf}(A \otimes B) = \mathrm{Pf}(A) \times \mathrm{Pf}(B)$ *and*

$$\mathrm{Pf}(A)^2 = \det(A).$$

Proof Let us first prove the first identity. Set $f_i = Be_i$ where as before, $e_1, \cdots, e_n$ is an orthonormal basis of $\mathbb{R}^n$. Then

$$\begin{aligned}
e^{\frac{1}{2}\sum_{i,j=1}^n a_{ij} f_i \wedge f_j} &= e^{\frac{1}{2}\sum_{i,j=1}^n \sum_{k,l=1}^n b_{ki} a_{ij} b_{lj} e_k \wedge e_l} \\
&= e^{\frac{1}{2}\sum_{i,j=1}^n \sum_{k,l=1}^n (B^t AB)_{kl} e_k \wedge e_l}
\end{aligned}$$

hence

$$\mathrm{Pf}(B^t AB) = T\left(e^{\frac{1}{2}\sum_{i,j=1}^n a_{ij} f_i \wedge f_j}\right)$$

$$
\begin{aligned}
&= T\left(\frac{1}{2^k k!}\sum_{\sigma\in\Sigma_n}\epsilon(\sigma)a_{\sigma(1)\sigma(2)}a_{\sigma(3)\sigma(4)}\cdots a_{\sigma(2k-1)\sigma(2k)}f_1\wedge\cdots\wedge f_{2k}\right)\\
&= \det(B)\cdot T\left(\frac{1}{2^k k!}\sum_{\sigma\in\Sigma_n}\epsilon(\sigma)a_{\sigma(1)\sigma(2)}a_{\sigma(3)\sigma(4)}\cdots a_{\sigma(2k-1)\sigma(2k)}e_1\wedge\cdots\wedge e_{2k}\right)\\
&= \det(B)\cdot\frac{(-1)^k}{2^k k!}\sum_{\sigma\in\Sigma_n}\epsilon(\sigma)a_{\sigma(1)\sigma(2)}a_{\sigma(3)\sigma(4)}\cdots a_{\sigma(2k-1)\sigma(2k)}\\
&= \det(B)\cdot \mathrm{Pf}(A).
\end{aligned}
$$

The second point in the proposition follows since $B\in SO_n(\mathbb{R})\Rightarrow \det B=1, \quad \text{and} \quad B^t=B^{-1}$.

The multiplicative property $\mathrm{Pf}(A\otimes B)=\mathrm{Pf}(A)\times\mathrm{Pf}(B)$ is easy to check. Finally, the last point clearly holds true for the matrix given by

$$
\begin{matrix} 0 & -\theta \\ \theta & 0 \end{matrix}
$$

and hence for a tensor product of such matrices. Since any matrix $A\in so_n(\mathbb{R})$ is conjugate to such a tensor product, there is a matrix $B\in SO_n(\mathbb{R})$ such that $B^{-1}AB$ is of this product type. The result then follows from the invariance property of the Pfaffian we proved above. □

Corollary 2 *The Pfaffian*

$$
\begin{matrix} so(\mathbb{R}^n) & \to & \mathbb{R} \\ A & \mapsto \mathrm{Pf}(A) & \end{matrix}
$$

defines an $SO_n(\mathbb{R})$-invariant polynomial on $so_n(\mathbb{R})$.

5 From connections to Chern-Weil forms

Let $E\to M$ be a vector bundle equipped with a connection ∇^E. Recall that the corresponding curvature is defined by $\Omega^E=\nabla^E\nabla^E$ so that

$$
\Omega^E(U,V)=[\nabla^E_U,\nabla^E_V]-\nabla^E_{[U,V]}\quad \forall U,V\in C^\infty(M,TM).
$$

If $L\in\Omega(M,\mathrm{End}(E))$ then for any $\sigma\in C^\infty(M,E)$,

$$
\nabla^E(L(\sigma))=(\nabla^{\mathrm{End}(E)}L)\sigma+(-1)^{|L|}L\nabla\sigma
$$

where $|L|$ is the degree of the $\mathrm{End}(E)$ valued form L and $\nabla^{\mathrm{Hom}(R)}$ the connection induced by ∇^E on $\mathrm{End}(E)$. As a consequence we have can write $\nabla^{\mathrm{End}(E)}L=[\nabla^E,L]$ where the bracket is a graded bracket, i.e. $[\nabla^E,L]=\nabla^E L-L\nabla^E$ if L is even and $[\nabla^E,L]=\nabla^E L+L\nabla^E$ if L is odd.

Lemma 3 *Let G be the structure group of E, let P be a G-invariant polynomial and p the associated polarized polynomial. Then, for any $\alpha_i \in \Omega(\mathrm{End}(E))$:*

$$dp(\alpha_1, \cdots, \alpha_p) = p([\nabla^E, \alpha_1], \alpha_2, \cdots, \alpha_p) + \cdots + p([\alpha_1, \alpha_2, \cdots, [\nabla^E, \alpha_p]).$$

Prooof Write $\nabla^E = d + \theta^E$ in a local triviallisation of E. For any $\alpha \in \Omega(M, \mathrm{End}(E))$, using the invariance, the symmetry and the multi-linearity of p we have:

$$\begin{aligned} dp(\alpha_1, \cdots, \alpha_p) &= p(d\alpha_1, \alpha_2, \cdots, \alpha_p) + p(\alpha_1, d\alpha_2, \cdots, \alpha_p) + \cdots + p(\alpha_1, \cdots, d\alpha_p) \\ &= p(d\alpha_1, \alpha_2, \cdots, \alpha_p) + p([\theta^E, \alpha_1], \alpha, \cdots, \alpha_p) + \cdots \\ &+ \cdots + p(\alpha_1, \alpha_2, \cdots, d\alpha_p) + p(\alpha_1, \alpha, \cdots, [\theta^E, \alpha_p]) \\ &= p([\nabla^E, \alpha_1], \alpha_2, \cdots, \alpha_p) + \cdots + p([\alpha_1, \alpha_2, \cdots, [\nabla^E, \alpha_p]). \end{aligned}$$

□

Lemma 4

- $\Omega^E \in \Omega^2(M, \mathrm{End}(E))$
- *If E is a real oriented vector bundle equipped with a metric $\langle \cdot, \cdot \rangle$ (in which case its structure group reduces to $G = SO(V)$ where V is the model space for E) and ∇^E is compatible with the metric i.e.*

  $$d\langle \cdot, \cdot \rangle = \langle \nabla^E \cdot, \cdot \rangle + \langle \cdot, \nabla^E \cdot \rangle$$

  then $\Omega^E \in \Omega^2(M, so(E))$ where $so(E)$ is the sub-bundle of antisymmetric morphisms of E.

Proof

- Let us check that the curvature is tensorial from which it will follow that it is a $\mathrm{End}(E)$-valued two form on M. Let $\sigma \in C^\infty(M, E)$ and $f \in C^\infty(M, \mathbb{C})$, we have:

  $$\begin{aligned} \Omega^E(U,V) f\sigma &= [\nabla_U, \nabla_V] f\sigma - \nabla_{[U,V]} f\sigma \\ &= ([U,V]f)\sigma - (\nabla_{[U,V]} f)\sigma + f\Omega^E(U,V)\sigma \\ &= f\Omega^E(U,V)\sigma. \end{aligned}$$

- Let $\sigma, \rho \in C^\infty(M, E)$ and $U, V \in C^\infty(M, TM)$, then

  $$\begin{aligned} 0 &= (UV - VU - [U,V])\langle \sigma, \rho \rangle \\ &= U\langle \nabla^E_V \sigma, \rho \rangle + U\langle \sigma, \nabla^E_V \rho \rangle \\ &- V\langle \nabla^E_U \sigma, \rho \rangle - V\langle \sigma, \nabla^E_U \rho \rangle \\ &- \langle \nabla^E_{[U,V]} \sigma, \rho \rangle - \langle \sigma, \nabla^E_{[U,V]} \rho \rangle \\ &= \langle \nabla^E_U \nabla^E_V \sigma, \rho \rangle + \langle \nabla^E_V \sigma, \nabla^E_U \rho \rangle \\ &+ \langle \nabla^E_U \sigma, \nabla^E_V \rho \rangle + \langle \sigma, \nabla^E_U \nabla^E_V \rho \rangle \\ &- \langle \nabla^E_V \nabla^E_U \sigma, \nabla^E_V \rho \rangle - \langle \sigma, \nabla^E_U \nabla^E_V \rho \rangle \\ &- \langle \nabla^E_V \sigma, \nabla^E_V \rho \rangle - \langle \sigma, \nabla^E_V \nabla^E_U \rho \rangle \\ &- \langle \nabla^E_{[U,V]} \sigma, \rho \rangle - \langle \sigma, \nabla^E_{[U,V]} \rho \rangle \\ &= \langle \Omega^E(U,V)\sigma, \rho \rangle + \langle \sigma, \Omega^E(U,V)\rho \rangle \end{aligned}$$

so that $\langle \Omega^E(U,V)\sigma, \rho\rangle = -\langle \sigma, \Omega^E U, V\rho\rangle$ which shows that $\Omega^E(U,V)$ is antisymmetric.

□

Proposition 3 *Let $E \to M$ be a vector bundle with structure group G equipped with a connection ∇^E. Let P be a homogeneous polynomial of degree k on $\mathfrak{g}$, then*

$$dP(\Omega^E) = 0,$$

and if $\nabla_t^E, t \in \mathbb{R}$ is a smooth family [1]*, then the following transgression formula holds:*

$$\frac{d}{dt} P(\Omega_t^E) = k \cdot dp(\dot{\nabla}_t^E, \Omega_t^E, \cdots, \Omega_t^E)$$

where $\dot{\nabla}_t^E$ is the derivative of ∇_t^E in the Fréchet topology, and Ω_t^E the curvature of ∇_t^E. In particular, the de Rham class of $P(\Omega^E)$ is independent of the choice of connection.

Proof To simplify notations, we momentarily drop the upper index E in the connection and the curvature.

- Applying the above lemma to $\alpha_i = \Omega \quad \forall i = 1, \cdots, n$ yields $dP(\Omega) = 0$ by the Bianchi identity.
- First observe that $\frac{d}{dt}\nabla_t \in \Omega^1(M, \mathrm{End}(E))$ and that

  $$\frac{d}{dt}\Omega_t = \frac{d}{dt}\nabla_t \nabla_t + \nabla_t \frac{d}{dt}\nabla_t = [\nabla_t, \frac{d}{dt}\nabla_t].$$

  Hence,

  $$\begin{aligned} \frac{d}{dt}P(\Omega_t) &= p(\frac{d}{dt}\Omega_t, \Omega_t, \cdots, \Omega_t) + p(\Omega_t, \frac{d}{dt}\Omega_t, \cdots, \Omega_t) + \cdots + p(\Omega_t, \cdots, \frac{d}{dt}\Omega_t) \\ &= k \cdot p(\frac{d}{dt}\Omega_t, \Omega, \cdots, \Omega) \\ &= k \cdot p([\nabla_t, \frac{d}{dt}\nabla_t], \Omega_t, \cdots, \Omega_t) \\ &= k \cdot dp(\frac{d}{dt}\nabla_t, \Omega_t, \cdots, \Omega_t) \end{aligned}$$

  where we applied the first lemma of this paragraph to $\alpha_1 = \frac{d}{dt}\nabla_t$ and $\alpha_i = \Omega$ using again the Bianchi identity.

□

An important property of invariant polynomials in view of what follows is their transformation property under diffeomorphisms. We have

$$P(f^*\alpha) = f^* P(\alpha)$$

for any diffeomorphism $f : N \to M$ of two smooth manifolds.
In particular, characteristic classes $c(\nabla^E)$ we are about to define given by de Rham class $P(\Omega^E)$ have a *naturality property*

$$c(f\nabla^{f^*E}) = f^* c(\nabla^E)$$

---

[1] here we are thinking of the space of smooth connections as a Fréchet space

which follows from the transformation property of the invariant polynomials under diffeomorphisms. Here $f : N \to M$ is a diffeomorphisms, $E \to M$ a vector bundle, $f^*E \to N$ its pull back to N, equipped respectively with a connection ∇^E and its pull-back $\nabla^{f*E} = f^*\nabla^E$.

Example 3 Chern character and first Chern class. *Given a function with Taylor expansion at all orders at* 0*, namely* $f(z) = \sum_{k=0}^K \frac{f^{(k)}}{k!}(0)z^k + o(z^K) \quad \forall K \in \mathbb{N}$ *and a complex vector bundle* (E, ∇^E) *with connection, it follows from Proposition 3 that* $P(\Omega) = \mathrm{tr} f(\Omega^E)$ *defines a closed form with cohomology class independent of the choice of connection. Choosing* $f = Id$ *yields the first Chern class*

$$c_1(\nabla^E) := \mathrm{tr}(\Omega^E).$$

Choosing $f(z) = e^{-z}$ *yields the Chern character*

$$\mathrm{ch}(\nabla^E) = \mathrm{tr}(e^{-\Omega^E}).$$

It obeys the following properties

$$\mathrm{ch}(\nabla^{E\oplus F}) = \mathrm{ch}(\nabla^E) + \mathrm{ch}(\nabla^F), \quad \mathrm{ch}(\nabla^{E\otimes F}) = \mathrm{ch}(\nabla^E) \wedge \mathrm{ch}(\nabla^F)$$

which justify the terminology "character".

Example 4 The Euler class, the $\hat{A}$-genus, the L genus. *Given a function with Taylor expansion at all orders at* 0*, namely* $f(z) = \sum_{k=0}^K \frac{f^{(k)}}{k!}(0)z^k + o(z^K) \quad \forall K \in \mathbb{N}$ *and an oriented metric real vector bundle* (E, ∇^E) *equipped with a connection compatible with the metric, it follows from Proposition 3 that* $P(\Omega) = \mathrm{Pf}\left(f(\Omega^E)\right)$ *defines a closed form with cohomology class independent of the choice of connection.*

Choosing $f(z) = -z$ *yields the Euler class*

$$e(\nabla^E) = \mathrm{Pf}(-\Omega^E) \in \Omega^N(M)$$

where N *is the rank of* E*. The Euler class vanishes if* N *is odd as a consequence of the vanishing of the Pfaffian in odd dimensions. Moreover, as a consequence of the multiplicativity of the Pfaffian on tensor products, this characteristic class obeys the following property:*

$$e(\nabla^{E\oplus F}) = e(\nabla^E) \wedge e(\nabla^F).$$

If M *is an oriented Riemannian surface, and* $(E = TM, \nabla^{TM})$ *is the tangent bundle equipped with the Levi-Civita connection, then*

$$e(\nabla^{TM}) = \kappa \quad d\mathrm{vol}$$

where κ *is the Gaussian curvature.*

Choosing $f(z) = \frac{\frac{z}{2}}{\mathrm{sh}\frac{z}{2}}$ *yields the* $\hat{A}$*-genus*

$$\hat{A}(\nabla^E) = \mathrm{Pf}\left(\frac{\frac{\Omega^E}{2}}{\mathrm{sh}\frac{\Omega^E}{2}}\right)$$

and $f(z) = \frac{\frac{z}{2}}{\frac{\mathrm{th}\, z}{2}}$ *the L-genus*

$$L(\nabla^E) = \mathrm{Pf}\left(\frac{\frac{\Omega^E}{2}}{\mathrm{th}\frac{\Omega^E}{2}}\right).$$

As a consequence of the multiplicativity of the Pfaffian on tensor products, these characteristic classes obey the following property:

$$\hat{A}(\nabla^{E\oplus F}) = \hat{A}(\nabla^E) \wedge \hat{A}(\nabla^F); \quad L(\nabla^{E\oplus F}) = L(\nabla^E) \wedge L(\nabla^F).$$

6 GeneralizedLaplacians

6.1 Differential operators on sections of vector bundles

The notion of differential operator we introduced on open subsets of $\mathbb{R}^n$ extends to manifolds.

Let K be $\mathbb{R}$ or $\mathbb{C}$. Let

$$A = \sum_{|\alpha|\leq m} a_\alpha(\tilde{x}) \frac{\partial^{|\alpha|}}{\partial x^\alpha} \tag{2}$$

with $a_\alpha \in C^\infty(U, \mathcal{M}_{p,q}(K))$ be a differential operator on an open subset U of $\mathbb{R}^n$ where $\mathcal{M}_{p,q}(K)$ is the space of (p,q) matrices with coefficients in K. A diffeomorphism $\tilde{x} = \tilde{x}(x)$ of U induces the following transformation on TU:

$$\frac{\partial}{\partial x_j} = \sum_{k=1}^{n} \frac{\partial \tilde{x}_k}{\partial x_j} \frac{\partial}{\partial \tilde{x}_k}.$$

As a consequence, in the new coordinates $\tilde{x}$, the operator A reads

$$A = \sum_{|\alpha|\leq m} \tilde{a}_\alpha(\tilde{x}) \frac{\partial^{|\alpha|}}{\partial x^\alpha}$$

for some other (q,p) matrix $\tilde{a}_\alpha$ of smooth K-valued functions on U. It follows that the operator can be described in a similar way in these new coordinates. If we have $a_\alpha(x) = 0$ for some $|\alpha| = m$, then $\tilde{a}_\beta(\tilde{x}) = 0$ for some $|\beta| = m$ so that the order is conserved under a change of coordinates.

Given a smooth manifold M of dimension n, it therefore makes sense to define a differential operator $A : C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ of order m as a linear operator which has the above description (2) in any local chart of M since another local chart would give the same type of local description via a change of coordinates.

Let $GL_r(K)$ be the group of invertible K-valued $r \times r$ matrices. Given two maps $\tau_1 : U \to GL_p(K)$ and $\tau_2 : U \to GL_q(K)$, $\tau_2 A \tau_1$ defines another differential operator or order m on U since

$$\tau_2 A \tau_1 = \sum_{|\alpha|\leq m} \tau_2 a_\alpha(x) \tau_1 \frac{\partial^{|\alpha|}}{\partial x^\alpha},$$

which is of the same type as (2). Letting $\pi_E : E \to M$ and $\pi_F : F \to M$ be two K-vector bundles based on M of rank p, q respectively, this shows that the shape of the operator described in (2) is invariant under a change of triviallisation $\tau_E : U \to GL_p(K)$ on E and $\tau_F : U \to GL_q(K)$ on F.

It therefore makes sense to set the following definition.

Definition 17 *A differential operator of order m on a smooth n-dimensional manifold M is a linear map $A : C^\infty(M, E) \to C^\infty(M, F)$, where E, F are two K-vector bundles based on M of rank p, q respectively, such that each point x of M has a neighborhood U with coordinates $(x_1, \cdots, x_n)$ over which there is a local triviallisation $E_{|_U} \simeq U \times K^p$ and $F_{|_U} \simeq U \times K^q$ in which the operator A reads*

$$A = \sum_{|\alpha| \leq m} a_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}.$$

Given local trivialisations $E_{|_U} \simeq U \times K^p$, $F_{|_U} \simeq U \times K^q$, over an open subset $U \in M$, one can define the symbol $\sigma_A(x, \xi)$ of a differential operator $A : C^\infty(M, E) \to C^\infty(M, F)$ as the symbol of the corresponding operator $\tilde{A} : C^\infty(U, K^p) \to C^\infty(U, K^q)$ read in the local triviallisation. However this symbol changes when the triviallisation changes; it is not even invariant under changes of coordinates. Nevertheless, the leading symbol remains the same under changes of local coordinates on M; it only changes under a change of triviallisation of E and F by $\sigma_L \to D\sigma_L C$ where $C \in GL_q(K)$ and $D \in GL_p(K)$. Since invertibility is conserved under such a transformation, the notion of ellipticity carries out to differential operators acting on sections of vector bundles.

Definition 18 *An operator $A : C^\infty(M, E) \to C^\infty(M, F)$ is elliptic if in any local triviallisation its leading symbol $\sigma_L(A)(x, \xi)$ is invertible for any $\xi \neq 0$, $x \in M$.*

When $E = F$, we shall say that the leading symbol of an operator is scalar if $\sigma_L(x, \xi) = \alpha(x, \xi) \cdot Id$ where $\alpha(x, \xi)$ is a scalar function.

Since the symbol of an operator acting on functions on an open set is obtained "substituting" $-i\frac{\partial}{\partial x_j}$ by ξ_j in the expression of the operator, in practice, the leading symbol of a differential operator on a manifold is obtained "substituting" $i\xi_j$ to $\frac{\partial}{\partial x_j}$ in the leading part of the operator expressed in any local triviallisation.

6.2 Laplacians on vector bundles

Let (M, g) be a Riemannian manifold with Riemannian metric g. Here we take $E = F = M \times \mathbb{C}$. In a local chart, the metric in a local chart $(x_1, \cdots, x_n)$ reads $g(x) = g_{ij}(x) dx_i \otimes dx_j$ where g_{ij} is an $n \times n$ matrix; let (g^{ij}) denotes the inverse matrix of (g_{ij}) and $\det g$ the determinant of (g_{ij}). The Laplace-Beltrami operator is defined by

$$\begin{aligned} \Delta_g &:= -\frac{1}{\det g} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \sqrt{\det g} \cdot g^{ij} \frac{\partial}{\partial x_j} \\ &= -\sum_{i,j=1}^n g^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \text{terms of lower order.} \end{aligned}$$

As we will see later, this expression is independent of the choice of local chart. It has scalar leading symbol given by

$$\sigma_L(\Delta_g) = -\sum_{i,j=1}^{n} (-i)^2 \xi_i \xi_j g_{ij} = \|\xi\|^2.$$

It therefore defines an elliptic operator of order 2.

Notice that setting $g_{ij} = \delta_{ij}$ the Kronecker symbol of $\{i, j\}$ yields back the local expression of the Laplacian on $\mathbb{R}^n$.

Let $E \to M$ be a vector bundle based on a Riemannian manifold M and let it be equipped with a connection ∇^E. The Levi-Civita connection ∇ on M combined with ∇^E yields a connection $\nabla^{T^*M\otimes E}$ on $T^*M \otimes E$. Applied to a one form $\alpha \in \Omega(M, E)$ this connection reads $\nabla_X^{T^*M\otimes E}\alpha = \nabla_X^{Hom(TM,E)}\alpha = \nabla_X^E\alpha - \alpha(\nabla_X)$. Composed with ∇^E, this yields an operator $\nabla^{T^*M\otimes E}\nabla^E : C^\infty(M, E) \to C^\infty(M, T^*M \otimes T^*M \otimes E)$, or equivalently a bilinear form on $C^\infty(M, TM)$ with values in $C^\infty(\mathrm{End}(E))$:

$$\begin{aligned} \nabla^{T^*M\otimes E}\nabla^E\sigma(X, Y) &= \nabla_X^{T^*M\otimes E}\nabla_Y^E\sigma \\ &= \nabla_X^E\nabla_Y^E - \nabla_{\nabla_X Y}^E\sigma \quad \forall X, Y \in C^\infty(M, TM), \forall\sigma \in C^\infty(M, E). \end{aligned}$$

The trace of this bilinear form on TM yields a second-order differential operator

$$\begin{aligned} \Delta^E &:= -\mathrm{tr}(\nabla^{T^*M\otimes E}\nabla^E) \\ &= -\nabla_{e_i}^E\nabla_{e_i}^E - \nabla_{\nabla_{e_i}e_i}^E \end{aligned}$$

where $(e_i)_{i=1,\cdots,n}$ is an orthonormal basis of TM. This operator, called a (generalized) Laplacian on $C^\infty(M, E)$, is independent of the choice of basis. When $E := M \times \mathbb{C}$, it yields back the Laplace-Beltrami operator on $C^\infty(M,\mathbb{C})$ and we have:

$$\Delta_g = -\mathrm{tr}(\nabla df) = -\sum_{i,j=1}^{n}\left(g^{ij}(x)\partial_i\partial_j - \sum_{k=1}^{n}\Gamma_{ij}^k\partial_k\right)$$

where Γ_{ij}^k are the Christoffel symbols defined in a local system of coordinates $(x_1, \cdots, x_n)$ by $\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j} := \sum_{k=1}^n \Gamma_{ij}^k\frac{\partial}{\partial x_i}$. The first of these equalities shows that Δ_g is well-defined independently of the choice of local chart.

In local coordinates, the generalized Laplacian Δ^E reads:

$$\Delta^E = -\sum_{i,j=1}^{n} g^{ij}(x)\left(\nabla_{\frac{\partial}{\partial x_i}}^E\nabla_{\frac{\partial}{\partial x_j}}^E - \sum_{k=1}^{n}\Gamma_{ij}^k\nabla_{\frac{\partial}{\partial x_k}}^E\right)$$

where as before, the Γ_{ij}^k are the Christoffel symbols. Notice that since locally, we have $\nabla_{\frac{\partial}{\partial x_j}}^E = \frac{\partial}{\partial x_j} + \theta(\frac{\partial}{\partial x_j})$ where θ is $\mathrm{Hom}(E)$-valued a one form, the top order part of Δ^E coincides with the top order part of the Laplace-Beltrami operator. The leading symbol of a generalized Laplacian is therefore also given by:

$$\sigma_L(\Delta^E) = \sigma_L(\Delta_g) = \|\xi\|^2.$$

Definition 19 *A generalized Laplacian on a vector bundle $E \to M$ is a second order differential operator with scalar leading symbol given by $\|\xi\|^2$. It is therefore an elliptic differential operator.*

The Laplace-Beltrami operator on a Riemannian manifold and Laplacians on bundles provide examples of generalized Laplacians.

7 Dirac operators

Useful references are [BGV], [F], [LM].

7.1 The Dirac operator as a "root" of a generalized Laplacian

Definition 20 *A Dirac operator D on a ($\mathbb{Z}_2$-graded) vector bundle E ($E = E^+ \oplus E^-$) is a first order differential operator (of odd parity, i.e. $D(C^\infty(M, E^+)) \subset C^\infty(M, E^-)$, $D(C^\infty(M, E^-)) \subset C^\infty(M, E^+)$) such that D^2 is a generalized Laplacian.*

In particular, since the leading symbol is multiplicative i.e. $\sigma_L(AB) = \sigma_L(A)\sigma_L(B)$, its leading symbol should satisfy:

$$\sigma_L(D)^2(x, \xi) = \|\xi\|^2.$$

Looking for a differential operator $D = \sum_{i=1}^n c_i \frac{\partial}{\partial x_i}$ whose square is the Laplacian $\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial^2 x_i}$ on $\mathbb{R}^n$ lead Dirac to the following constraints on the coefficients c_i:

$$c_i c_j + c_j c_i = -2\delta_{ij}. \tag{3}$$

These relations correspond to the Clifford relations for the euclidean structure on $\mathbb{R}^n$ and generate the Clifford algebra on $\mathbb{R}^n$ which we are about to define.

Let K be $\mathbb{R}$ or $\mathbb{C}$. Let V be a K- vector space and q a quadratic form on V.

Definition 21

- *The Clifford algebra on (V, q) is the algebra over K generated by V with relations*

  $$v \cdot w + w \cdot v = -2q(v, w) \quad \forall v, w \in V$$

  (or equivalently $v \cdot v = -q(v) \quad \forall v \in V$.) Equivalently, is is the quotient of the tensor algebra $\mathcal{T}(V) = \sum_k V^{\otimes k}$ generated by V by the ideal $\mathcal{I}(V, q)$ generated by the set $\{v \cdot w + w \cdot v + 2q(v, w), \quad v, w \in V\}$.

- *A $Cl(V, q)$ - module (called a Clifford module when not specifiying which Clifford algebra) is a K-vector space E together with a representation $c : Cl(V, q) \to Hom_K(E)$, so that we have:*

$$c(v)c(w) + c(w)c(v) = -2q(v, w)Id_E, \quad \forall v, w \in V. \tag{4}$$

Example 5 *Let $V = \mathbb{R}^{r+s}$ wth $Q((x_1, \cdots, x_n)) = x_1^2 + \cdots x_r^2 - x_{r+1}^2 - \cdots x_{r+s}^2$ and set $Cl_{r,s}(\mathbb{R}) := Cl(V, q)$ then $Cl_{r,s}(\mathbb{R})$ is generated by a q-orthonormal basis $e_1, \cdots, e_{r+s}$ subject to the relations:*

$$e_i e_j + e_j e_i = -2\delta_{ij} \quad \text{if} \quad i \leq r, \quad e_i e_j + e_j e_i = -2\delta_{ij} \quad \text{if} \quad i > r.$$

When $r = n, s = 0$ we denote the corresponding Clifford algebra by $Cl_n(\mathbb{R})$. The relations which define $Cl_n(\mathbb{R})$ therefore correspond to equation (3). Notice that the action of the orthonormal group $O_n(\mathbb{R})$ on $\mathbb{R}^n$ extends to one on $CL_n(\mathbb{R})$ since the relations that define the Clifford algebra are invariant under this action.

The following universal property is very useful:

Lemma 5 *A is a K-algebra and $c : (V, q) \to A$ a map verifying*

$$c(v)c(w) + c(w)c(v) = -2q(v, w)1_A, \quad \forall v, w \in V,$$

(such a map is called a Clifford map) then there is a unique extension of c to a map $\tilde{c} : C(V, q) \to A$.

In practice, it is therefore often sufficient to define the Clifford map c on the vector space V only.

There is a natural $\mathbb{Z}_2$-grading on $Cl(V, q) = Cl^+(V, q) \oplus Cl^-(V, q)$ given by the natural grading on $\mathcal{T}(V) = \sum_{k=0}^{\infty} V^{\otimes 2k} \oplus \sum_{k=0}^{\infty} V^{\otimes 2k+1}$. It is clear that $V \subset Cl^-(V, q)$.

Definition 22 *A $\mathbb{Z}_2$-graded Clifford module is a Clifford module with a superspace structure $E = E^+ \oplus E^-$ such that the Clifford action is even with respect to this $\mathbb{Z}_2$-grading on E:*

$$Cl^+(V, q)E^+ \subset E^+; \quad Cl^+(V, q)E^- \subset E^-; Cl^-(V, q)E^+ \subset E^-; Cl^-(V, q)E^- \subset E^+.$$

7.2 From a Clifford connection to a Dirac operator

Useful references are [F], [LM].
If M is a Riemannian manifold, at each point $x \in M$, we have an inner product $q_x := \langle \cdot, \cdot \rangle_x$ on the cotangent space $T_x^* M$. Following the above construction we can build the Clifford algebra $Cl(T_x^* M, q_x)$. A Clifford bundle is a smooth family of such Clifford algebras.

Definition 23

- *If M is a Riemannian manifold, the Clifford bundle $Cl(M)$ is the bundle of Clifford algebras on M whose fibre at $x \in M$ is the Clifford algebra $Cl_x(M) := Cl(T_x^* M, q_x)$, with g_x the quadratic form given by the Riemannian structure on M.*
- *Equivalently, it is an associated bundle to the orthonormal frame bundle:*

$$Cl(M) = O(M) \otimes_{O(n)} Cl_n(\mathbb{R}).$$

Remark 6 *Notice that in local coordinates $(x_1, \cdots, x_n)$ on M:*

$$c(dx_i)c(dx_j) + c(dx_j)c(dx_i) = -2g_{ij}(x).$$

In particular, if $\xi \in T_x^* M$,then

$$\xi \cdot \xi = c(\xi)^2 = \frac{1}{2} \left(\sum_{i=1}^{n} \xi_i c(dx_i) \right) \cdot \left(\sum_{j=1}^{n} \xi_j c(dx_j) \right) = - \sum_{i,j=1}^{n} g_{ij}(x) \xi_i \cdot \xi_j = -\|\xi\|^2,$$

where we use the fact that $\langle dx_i, dx_j \rangle = \partial_{ij}$.

The Levi-Civita connection ∇ on M induces one on $O(M)$ and therefore a connection (also denoted by ∇) on $Cl(M)$ which is compatible with the Clifford product:

$$\nabla(ab) = (\nabla a)b + a(\nabla b), \quad \forall a, b \in C^\infty(M, Cl(M)).$$

Definition 24

- *A (graded) Clifford module E on a manifold M is a ($\mathbb{Z}_2$-graded) vector bundle $E \to M$ based on M with a (graded) action of the Clifford bundle $Cl(M)$ on it:*

$$c : Cl(M) \to \mathrm{End}(E)$$

- *A Clifford connection on a Clifford module E is a connection ∇^E on E which is compatible with the Clifford action:*

$$[\nabla^E, c(a)] = c(\nabla a), \quad \forall a \in C^\infty(M, Cl(M)).$$

Proposition 4 *Let M be a Riemannian manifold and E a Clifford module on M (with Clifford multiplication c) equipped with a Clifford connection ∇^E. The first order differential operator $D : C^\infty(M, E) \to Ci(M, E)$ defined by $D = c \circ \nabla^E$ or in local coordinates $(x_1, \cdots, x_n)$:*

$$D = \sum_{i=1}^n c(dx_i) \nabla^E_{\frac{\partial}{\partial x_i}}$$

has leading symbol

$$\sigma_L(D)(x, \xi) = i\xi.$$

It is therefore a Dirac operator, i.e its square D^2 is a generalized Laplacian.

Proof The leading symbol of D is obtained replacing $\frac{\partial}{\partial x_j}$ by $i\xi_j$,

$$\sigma_L(D)(x, \xi) = i \sum_{j=1}^n c(dx_j)\xi_j = i \sum_{j=1}^n dx_j \xi_j = i\xi.$$

Hence

$$\sigma_L(D)^2(x, \xi) = -\xi \cdot \xi = \|\xi\|^2$$

where $\cdot$ stands for the Clifford multiplication and where we have used the Clifford relations for the last equality. $\square$

8 $d + d^*$ as a Dirac operator on the Clifford module of differential forms

Useful references are [BGV], [G], [LM].

8.1 A Clifford multiplication on forms

As the following lemma shows, combining the interior and exterior products on $\Omega(M)$, where M is a Riemannian manifold, yields a Clifford multiplication from which one can build a Dirac operator using the Levi-Civita connection.

Lemma 6

- $\epsilon(v)^* = i(v) \quad \forall v \in T_xM, x \in M,$

- *$c = \epsilon - i$ defines a Clifford multiplication on $\Omega(M)$, i.e.*

  $$c(v)c(w) + c(w)c(v) = -2\langle v, w\rangle_x, \quad \forall v, w \in T_xM$$

  where $\langle \cdot, \cdot, \rangle_x$ is the inner product on T_xM induced by the metric structure.

- *$d = \sum_{i=1}^n \epsilon(e_j)\nabla_{e_j}$,*

- *$d^* = -\sum_{j=1}^n i(e_j)\nabla_{e_j}$, where $(e_1, \cdots, e_n)$ is an orthonormal basis of T_xM.*

(Partial) Proof To avoid technicalities, we prove the results on one forms only.

- Given $v \in T_xM$, $f \in \Omega^0(M)$ and $\alpha \in T_x^*M$ we have:

  $$\langle i(v)\alpha, f(x)\rangle_x = \langle \alpha(v), f(x)\rangle_x = \alpha(v)f(x).$$

  On the other hand

  $$\langle \alpha, \epsilon(v)f(x)\rangle_x = \langle \alpha, f(x)v^\flat\rangle = \langle \alpha(x), v^\flat\rangle_x f(x) = \alpha(v)\, f(x).$$

  Hence $\epsilon^* = i$ on 1-forms.

- Let $v, w \in T_xM$. First observe that

  $$\epsilon(v)i(w) + i(w)\epsilon(v) = \langle v, w\rangle_x \quad \forall v, w \in T_xM.$$

  Here again, we check the property on one a 1-form α.

  $(\epsilon(v)i(w) + i(w)\epsilon(v))\,\alpha = \alpha(w)v^\flat + i(w)(v^\flat \wedge \alpha) = \alpha(w)v^\flat + v^\flat(w)\alpha - v^\flat\alpha(w) = v\flat(w)\alpha = \langle v, w\rangle_x\,\alpha.$

  As a consequence we have:

  $c(v)c(w)+c(w)c(v) = \epsilon(v)\epsilon(w)+\epsilon(w)\epsilon(v)+i(v)i(w)+i(w)i(v)-2\left(\epsilon(v)i(w) + i(w)\epsilon(v)\right) = -2\langle v, w\rangle_x,$

  where we have used the fact that $\epsilon(v)\epsilon(w) + \epsilon(w)\epsilon(v) = i(v)i(w) + i(w)i(v) = 0$.

- Let us set $\tilde{d} = \sum_{i=1}^n \epsilon(e_j)\nabla_{e_j}$ and show that $\tilde{d}$ satisfies the requirements i), ii), iii) which define d uniquely:

  i) Since ∇_{e_j} sends $\Omega^p(M)$ to $\Omega^p(M)$, and $\epsilon(e_j)$ increases the degree of the form by 1, $\tilde{d}$ sends $\Omega^p(M)$ to $\Omega^{p+1}(M)$.

  ii) $\tilde{d} \circ \tilde{d}(f) = 0 \quad \forall f \in C^\infty(M,\mathbb{C})$. We prove that $\tilde{d}^2 f = -\langle T, df\rangle$ where T is the torsion. Since the torsion of the Levi-Civita connection vanishes by definition, this will prove that $\tilde{d}^2 = 0$. To simplify notations we set $\nabla_j = \nabla_{\frac{\partial}{\partial x_j}} = \nabla_{e_j}$ where $e_j = \frac{\partial}{\partial x_j}$.

  $$\begin{aligned} \tilde{d}^2 f &= \tilde{d}(df) \\ &= \sum_{ij} \epsilon(dx_i)\nabla_i(\partial_j f dx_j) \\ &= \sum_{ij} \partial_i\partial_j f dx_i \wedge dx_j + \sum_{ij} \epsilon(dx_i)\partial_j f \nabla_i(dx_j) \\ &= \sum_{ij} \epsilon(dx_i)\partial_j f \nabla_i(dx_j). \end{aligned}$$

By Leibniz's rule:

$$0 = \frac{\partial}{\partial x_i}\langle dx_j, e_k\rangle = \langle \nabla_i dx_j, e_k\rangle + \langle dx_j, \nabla_i e_k\rangle$$

so that

$$\begin{aligned}
\tilde{d}^2 f &= -\sum_{ijk} \epsilon(dx_i)\frac{\partial}{\partial x_j} f\langle dx_j, \nabla_i e_k\rangle dx_k \\
&= -\sum_{ijk} \frac{\partial}{\partial x_j} f\langle dx_j, \nabla_i e_k\rangle_x \, dx_i \wedge dx_k \\
&= -\sum_{i<k} \langle df, \nabla_i e_k - \nabla_k e_i\rangle_x \, dx_i \wedge dx_k \\
&= -\sum_{i<k} \langle df, T(e_i, e_k)\rangle_x \, dx_i \wedge dx_k \\
&= -\langle T, df\rangle.
\end{aligned}$$

iii) $\tilde{d}$ is a derivation. Indeed, the Levi-Civita connection on the tangent bundle TM extends to a connection on the exterior cotangent bundle ΛT^*M and satisfies the following rule:

$$\nabla_X(\alpha \wedge \beta) = \nabla_X \alpha \wedge \beta + \alpha \wedge \nabla_X \beta \quad \forall \alpha, \beta \in \Omega(M), \forall X \in C^\infty(M, TM).$$

Hence $\tilde{d} = \sum_i \epsilon(e_i^*)\nabla_{e_i}$ satisfies a graded Leibniz rule:

$$\tilde{d}(\alpha \wedge \beta) = \tilde{d}\alpha \wedge \beta + (-1)^{|\alpha|}\, \alpha \wedge \tilde{d}\beta \quad \forall \alpha, \beta \in \Omega(M)$$

and therefore yields a (graded) derivation.

- Given $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^{p+1}(M)$ we want to check that $\langle \epsilon(dx_i)\nabla_i \alpha, \beta\rangle = \langle \alpha, i(dx_i)\nabla_i \beta\rangle$. Differentiating the one form defined on $v \in T_xM$ by $\alpha(v) = \langle \alpha, i(v)\beta\rangle_x$ and using Leibniz's rule yields:

$$\sum_i (e_i\alpha(e_i) - \alpha(\nabla_i e_i)) = \langle \nabla_i \alpha, i(e_i)\beta\rangle_x + \langle \alpha, \nabla_i i(e_i)\beta\rangle_x = \langle \epsilon(e_i)\nabla_i \alpha, \beta\rangle_x + \langle \alpha, i(e_i)\nabla_i \beta\rangle$$

where we have used the fact that $\epsilon^* = i$. On the other hand since the divergence is given by $d^*\alpha = -\mathrm{tr}(\nabla\alpha)$ for a one form α, it follows from Stokes's theorem that $\mathrm{tr}(\nabla\alpha) := \sum_{i=1}^n \nabla\alpha(e_i, e_i) = \sum_i (e_i\alpha(e_i) - \alpha(\nabla_i e_i))$ integrates to 0 on M, i.e.

$$\int_M \mathrm{tr}(\nabla\alpha)\, d\mathrm{vol} = -\int_M d^*\alpha = 0.$$

Thus

$$\langle \epsilon(e_i)\nabla_i \alpha, \beta\rangle_x + \langle \alpha, i(e_i)\nabla_i \beta\rangle = 0$$

so that $d^* = -i \circ \nabla$.

□

8.2 A Dirac operator on forms

Proposition 5 *The operator* $D := d + d^* = \sum_{j=1}^{n} c(dx_j)\nabla_{\frac{d}{dx_i}}$ *is a Dirac operator and has leading symbol given by* $\sigma_L(D)(x,\xi) = i\xi$*. In particular, its square* $(d+d^*)^2$ *is a generalized Laplacian.*

Remark 7 *We leave it to the reader to check that this definition is independent of the chosen local coordinate chart* $(x_1, \cdots, x_n)$*.*

Proof This follows from the above lemma combined with Proposition 4.

The following proposition relates the *Hodge Laplacian* $\Delta := (d+d^*)^2$ with the generalized Laplacian $\Delta^{\Lambda T^*M}$ thus providing another proof that it is a generalized Laplacian. $\square$

Proposition 6 *Let* $\alpha \in \Omega(M)$ *then*

$$(d+d^*)^2(\alpha) = \Delta^{\Lambda T^*M}\alpha + \sum_{i<j} c(dx_i)c(dx_j)R(e_i,e_j)(\alpha)$$

where $R(u,v) := [\nabla_u, \nabla_v] - \nabla_{[u,v]}$ *is the curvature tensor.*

Proof First notice that if as before ∇ denotes the Levi-Civita connection on TM with Christoffel coefficients give by Γ_{ij}^k then the induced dual connection ∇^* on T^*M reads $\nabla_i^* dx_j = -\Gamma_{ik}^j dx_k$ for we have

$$\begin{aligned}
\langle \nabla_i^* dx_j, \frac{d}{dx_k}\rangle &= -\langle dx_j, \nabla_i \frac{d}{dx_k}\rangle \\
&= -\langle dx_j, \Gamma_{ik}^l \frac{d}{dx_l}\rangle = -\Gamma_{ik}^l \langle dx_j, \frac{d}{dx_l}\rangle \\
&= -\Gamma_{ik}^j.
\end{aligned}$$

(In the following, we drop the $\star$ in ∇^*). It follows that

$$\begin{aligned}
(d+d^*)^2\alpha &= \sum_{i,j=1}^{n} c(dx_i)\nabla_i(c(dx_j)\nabla_j\alpha) \\
&= \sum_{i,j=1}^{n} c(dx_i)c(\nabla_i dx_j)\nabla_j\alpha + \sum_{i,j=1}^{n} c(dx_i)c(dx_j)\nabla_i(\nabla_j\alpha) \\
&= -\sum_{i,j=1}^{n} c(dx_i)c(\Gamma_{ik}^j dx_k)\nabla_j\alpha \\
&- \sum_{i=1}^{n}\nabla_i\nabla_i\alpha + \sum_{i<j}^{n} c(dx_i)c(dx_j)\left(\nabla_i\nabla_j - \nabla_j\nabla_i\right)\alpha \\
&= -\sum_{i,j=1}^{n} c(dx_i)c(dx_k)\nabla_{\Gamma_{ik}^j \frac{d}{dx_j}}\alpha
\end{aligned}$$

$$
\begin{aligned}
- & \sum_{i=1}^{n} \nabla_i \nabla_i \alpha + \sum_{i<j} c(dx_i)c(dx_j)R(e_i,e_j)\alpha. \\
= & -\sum_{i<j} (c(dx_i)c(dx_k) + c(dx_k)c(dx_i)) \nabla_{\Gamma_{ik}^{j} \frac{d}{dx_j}} \alpha - \sum_{i=1}^{n} c(dx_i)^2 \nabla_{\nabla_i \frac{d}{dx_j}} \alpha \\
- & \sum_{i=1}^{n} \nabla_i \nabla_i \alpha + \sum_{i<j} c(dx_i)c(dx_j)R(e_i,e_j)\alpha \\
= & \Delta^{\Lambda T^* M} \alpha + \sum_{i<j} c(dx_i)c(dx_j)R(e_i,e_j)\alpha.
\end{aligned}
$$

□

We give another expression of the Hodge Laplacian on p-forms in terms of d and the Hodge star operator $*$ introduced previously.

Lemma 7 *The formal adjoint* $d^* : \Omega^p(M) \to \Omega^{p-1}(M)$ *of* d *is given by*

$$d^* = (-1)^{np+1} \star d \star .$$

Proof

$$
\begin{aligned}
\langle \alpha, d^* \beta \rangle &= \int_M \alpha \wedge d^* \beta \\
&= \int_M d\alpha \wedge \star\beta \\
&= \int_M d(\alpha \wedge \star\beta) - (-1)^p \int_M \alpha \wedge d \star \beta \\
&= (-1)^{p+1} \int_M \alpha \wedge d \star \beta \\
& \quad \left(\text{since} \int_M d\gamma = 0 \right) \\
&= (-1)^{p+1} (-1)^{(n-p)(n-(n-p))} \int_M \alpha \wedge \star\star d \star \beta \\
&= (-1)^{np+1} \int_M \alpha \wedge \star(\star d \star \beta) \\
&= (-1)^{np+1} \langle \alpha, \star d \star \beta \rangle
\end{aligned}
$$

□

The following proposition follows in a straightforward way from the above lemma.

Proposition 7 *On* $\Omega^p(M)$ *we have:*

$$\Delta = (-1)^{n(p+1)+1} \star d \star \circ d + (-1)^{np+1} d \star d \star .$$

9 Two index theorems

We give two index theorems for the Dirac operator $D = d + d^*$ where d is the exterior differential on forms on some closed Riemannian manifold M. In both cases the Dirac operator is acting on $\Omega(M)$ but the $\mathbb{Z}_2$-grading on that space changes. In the first case, the space splits into even and odd forms thus leading to the following index theorem:

$$\mathrm{ind}(D) = \chi(M) = \frac{1}{\sqrt{2\pi}^n}\int_M e(\nabla^{TM}),$$

where $\chi(M)$ is the Euler characteristic. In the second case, the $\mathbb{Z}_2$-grading is induced by the chirality operator, thus leading to the following index theorem when the dimension of M is a multiple of 4:

$$\mathrm{ind}(D) = \sigma(M) = \frac{(-1)^l}{\pi^{2l}}\int_M L(\nabla^{TM}),$$

where $\sigma(M)$ is the signature operator, and $n = 4l$ is the dimension of the manifold.

The space of forms $\Omega(M)$ splits $\Omega(M) = \Omega^+(M) \oplus \Omega^-(M)$ in two ways,

- letting $\Omega^+(M)$ be the space of even degree forms, and $\Omega^-(M)$ be the set of odd degree forms,
- letting $\Omega^+(M)$ be the space of self-dual forms and $\Omega^-(M)$ the space of anti self-dual forms.

9.1 A first chiral Dirac operator on $\Omega(M)$

Let us describe the first splitting. Given a vector space V, we define a parity operator on ΛV by:

$$\begin{aligned} P : \Lambda V &\to \Lambda V \\ \alpha &\mapsto (-1)^{|\alpha|}\alpha \end{aligned}$$

which satisfies $P^2 = I$. It induces a bundle morphism on ΛT^*M defined on a fibre above $x \in M$ by:

$$\begin{aligned} P_x : \Lambda T_x^*M &\to \Lambda T_x^*M \\ \alpha(x) &\mapsto (-1)^{|\alpha(x)|}\alpha(x). \end{aligned}$$

The operator P induces the following splitting of the space of forms:

$$\Omega(M) = \Omega^{ev}(M) \oplus \Omega^{odd}(M)$$

where we have set:

$$\Omega^{ev}(M) := \{\alpha \in \Omega(M), |\alpha| \in 2\mathbb{N}\} = \mathrm{Ker}(P - I)$$

and

$$\Omega^{odd}(M) := \{\alpha \in \Omega(M), |\alpha| \in 2\mathbb{N} + 1\} = \mathrm{Ker}(P + I).$$

Since $D = d + d^*$ increases or decreases the degree by 1, it anti-commutes with P and therefore sends $\Omega^{ev}(M)$ to $\Omega^{odd}(M)$ and vice versa. This gives rise to a chiral Dirac operator:

$$D_P^+ := D_{|_{\Omega^{ev}(M)}} : \Omega^{ev}(M) \to \Omega^{odd}(M).$$

In order to describe the second splitting, we first introduce introduce the chirality operator Γ which will play the role of the parity operator P in the above. Proving that it anti-commutes with D will take us a while.

9.2 The chirality operator

Let (V, q) be an euclidean space, and let $Cl(V) \otimes \mathbb{C}$ denote the associated complexified Clifford algebra. Given an orthonormal basis $\{e_1, \cdots, e_n\}$ of V, the *chirality operator* $\Gamma \in C(V) \otimes \mathbb{C}$ is defined by:

$$\Gamma := i^{k(n)} e_1 \cdot e_2 \cdots e_n$$

where the multiplication is the Clifford multiplication and $k(n) = \frac{n}{2}$ if n is even, $k(n) = \frac{n+1}{2}$ if n is odd.

It acts on $\Lambda V^{\mathbb{C}}$ where $V^{\mathbb{C}}$ is the complexification of V.

Lemma 8
- *Γ does not depend on the chosen orthonormal basis,*
- *$\Gamma^2 = 1$.*

Proof We only prove the second point, leaving the first point as an exercise.

$$\begin{aligned}
\Gamma^2 &= \left(i^{k(n)}\right)^2 e_1 \cdots e_n \cdot e_1 \cdots e_n \\
&= (-1)^{k(n)+n-1} e_1 \cdots e_n \cdot e_2 \cdots e_n \\
&= (-1)^{k(n)+\sum_{j=1}^{n-1} j} e_1^2 \cdots e_2^2 \cdots e_n^2 \\
&= (-1)^{k(n)+\frac{n(n-1)}{2}} (-1)^n \\
&= 1 \quad \text{if} \quad n = 2k \quad \text{or} \quad n = 2k+1.
\end{aligned}$$

□

As a consequence of this lemma, if M has even dimension, Γ provides a grading of the (complex) Clifford module $\Omega_{\mathbb{C}}(M) := C^\infty(M, \Lambda T^*M \otimes \mathbb{C}) = \Omega_{\mathbb{C}}^+(M) \oplus \Omega_{\mathbb{C}}^-(M)$ where

$$\Omega_{\mathbb{C}}^+(M) := \{\alpha \in \Omega_{\mathbb{C}}(M), \Gamma(\alpha) = \alpha\} = \mathrm{Ker}(\Gamma - I)$$

and

$$\Omega_{\mathbb{C}}^-(M) := \{\alpha \in \Omega_{\mathbb{C}}(M), \Gamma(\alpha) = -\alpha\} = \mathrm{Ker}(\Gamma + I).$$

Proposition 8 *Given a Riemannian manifold (M, g), the chiral operator defined fibrewise by*

$$\begin{aligned}
\Gamma_x : \Lambda^p T_x^* M \otimes \mathbb{C} &\to \Lambda^{n-p} T_x^* M \otimes \mathbb{C} \\
\alpha(x) &\mapsto i^{k(n)}\, c(e_1^*(x)) \cdots c(e_n^*(x)) \alpha(x)
\end{aligned}$$

where $e_i^*(x)$ *is an orthonormal basis of* T_x^*M *does not depend on the choice of orthonormal basis.* Γ *satisfies the following properties:*

- *On any smooth function* $f \in \Omega^0(M)$,

$$\Gamma f = i^{k(n)} f e_1^* \wedge \cdots \wedge e_n^*$$

  so that $\Gamma f = i^{k(n)} f$.

- *For any one form* $\alpha \in \Omega^1(M)$

$$\epsilon(\alpha)\Gamma = (-1)^n \Gamma i(\alpha). \tag{5}$$

- *Given* $\alpha \in \Omega^p(M), \beta \in \Omega^p(M)$,

$$\int_M \alpha \wedge \Gamma\beta = (-1)^{pn+\frac{p(p-1)}{2}} i^{k(n)} \int_M \langle \alpha, \beta \rangle_x \mathrm{dvol}(x) \tag{6}$$

- *On* $\Omega^p(M)$ *we have:*

$$\Gamma = (-1)^{pn+\frac{p(p-1)}{2}} i^{k(n)} \star \tag{7}$$

- *The (formal) adjoint* d^* *of* d *reads:*

$$d^* = (-1)^{n+1}\Gamma d \Gamma.$$

- *In even dimensions,* $d + d^*$ *anti commutes with* Γ*:*

$$\Gamma(d + d^*) = -(d + d^*)\,\Gamma.$$

Proof

-

$$\begin{array}{rcl} e_1^* \cdots e_n^* \Gamma f & = & c(e_1^*) \cdots c(e_n^*) f \\ & = & c(e_1^*) \cdots (\epsilon(e_{n-1}^*) - i(\epsilon_{n-1}^*))\epsilon(e_n^*) f \\ & = & c(e_1^*) \cdots (e_{n-1}^* \wedge e_n^* f - 0) \\ & = & f e_1^* \wedge \cdots \wedge e_{n-1}^* \wedge e_n^* \end{array}$$

  so that $\Gamma f = i^{k(n)} f e_1^* \cdots e_n^*$.

- It is sufficient to prove the result for $\alpha = e_n^*$. On one hand, we have, for any $v \in T_xM$, $\alpha \in T_x^*M$:

$$\begin{array}{rcl} \epsilon(v)c(v)\alpha & = & \epsilon(v)(\epsilon(v) - i(v))\alpha \\ & = & \epsilon(v)(v^\flat \wedge \alpha - \alpha(v)) \\ & = & v^\flat \wedge v^\flat \wedge \alpha - \alpha(v) v^* \\ & = & -\alpha(v) v^\flat \wedge \alpha \\ & = & -(\epsilon(v) - i(v))\alpha(v) \\ & = & -(\epsilon(v) - i(v)) i(v)\alpha \\ & = & -c(v) i(v) \alpha. \end{array}$$

On the other hand, for $v \neq w \in T_xM$,$\alpha \in T_x^*M$, we have:

$$\begin{aligned}\epsilon(v)c(w)\alpha &= \epsilon(v)(\epsilon(w)-i(w))\alpha\\ &= \epsilon(v)(w\wedge\alpha-\alpha(w))\\ &= v\wedge w\wedge\alpha-\alpha(w)v\\ &= -w\wedge v\wedge\alpha+\epsilon(v)\alpha(w)\\ &= -(\epsilon(w)-i(w))\epsilon(v)\alpha\\ &= -c(w)\epsilon(v)\alpha.\end{aligned}$$

It follows that

$$\epsilon(e_n)c(e_1)\cdots c(e_n) = (-1)^{n-1}c(e_1)\cdots c(e_{n-1})\epsilon(e_n)c(e_n) = (-1)^n c(e_1)\cdots c(e_{n-1})i(e_n).$$

Hence $\epsilon(e_n)\Gamma = (-i)^n\Gamma i(e_n)$.

- The third point follows by induction. The formula holds for functions as we saw earlier in this proof. Assume it holds for p-forms and let $\alpha \in \Omega^p(M), \beta \in \Omega^{p+1}(M)$, $e \in \Omega^1(M)$, then:

$$\begin{aligned}\alpha\wedge e\wedge\Gamma\beta &= \alpha\wedge\epsilon(e)\Gamma\beta\\ &= (-1)^n\alpha\wedge\Gamma i(e)\beta \quad \text{using} \quad (5)\\ &= (-1)^{n+pn+\frac{p(p-1)}{2}}i^{k(n)}\langle\alpha, i(e)\beta\rangle e_1\wedge\cdots\wedge e_n \quad \text{by} \quad \text{assumption}\\ &= (-1)^{n(p+1)+\frac{p(p-1)}{2}}i^{k(n)}\langle\epsilon(e)\alpha,\beta\rangle e_1\wedge\cdots\wedge e_n \quad \text{since} \quad \epsilon(v)^* = i(v)\\ &= (-1)^{n(p+1)+\frac{p(p-1)}{2}+p}i^{k(n)}\langle\alpha\wedge e,\beta\rangle e_1\wedge\cdots\wedge e_n\\ &= (-1)^{n(p+1)+\frac{p(p+1)}{2}}i^{k(n)}\langle\alpha\wedge e,\beta\rangle e_1\wedge\cdots\wedge e_n.\end{aligned}$$

- On $\Omega^p(M)$, Γ is characterised by

$$\alpha\wedge\Gamma\beta = (-1)^{pn+\frac{p(p-1)}{2}}i^{k(n)}\langle\alpha,\beta\rangle_x d\text{vol}(x)$$

whereas $\star$ is characterised by

$$\alpha\wedge\star\beta = \langle\alpha,\beta\rangle_x d\text{vol}(x)$$

so that Γ and $\star$ are related by:

$$\Gamma = (-1)^{pn+\frac{p(p-1)}{2}}i^{k(n)}\star.$$

- To check that $d^* = (-1)^{n+1}\Gamma d\Gamma$, we use Stokes formula:

$$\begin{aligned}\langle\alpha, d^*\beta\rangle_g &= \int_M\langle\alpha, d^*\beta\rangle_x d\text{vol}(x)\\ &= \int_M\langle d\alpha,\beta\rangle_x d\text{vol}(x)\\ &= (-1)^{(p+1)n+\frac{p(p+1)}{2}}i^{-k(n)}\int_M d\alpha\wedge\Gamma\beta\end{aligned}$$

$$
\begin{aligned}
&= -(-1)^{p+(p+1)n+\frac{p(p+1)}{2}} i^{-k(n)} \int_M \alpha \wedge d\Gamma\beta \\
&= -(-1)^n \int_M \langle \alpha, \Gamma d\Gamma\beta \rangle d\mathrm{vol}(x) \\
&= \langle \alpha, (-1)^{n+1} \Gamma d\Gamma\beta \rangle_g
\end{aligned}
$$

- Thus, in even dimensions,

$$\Gamma(d + d^*) = \Gamma(d - \Gamma d\Gamma) = -(d - \Gamma d\Gamma)\Gamma = -(d + d^*)\Gamma.$$

□

As a consequence of this proposition, the Dirac operator $D = d + d^*$ anti-commutes with Γ. It therefore sends $\Omega_{\mathbb{C}}^+(M)$ to $\Omega_{\mathbb{C}}^-(M)$ and vice versa. We can therefore define a chiral Dirac operator, called the *signature operator*:

$$D_\Gamma^+ := D_{|_{\Omega_{\mathbb{C}}^+(M)}} : \Omega_{\mathbb{C}}^+(M) \to \Omega_{\mathbb{C}}^-(M).$$

9.3 The Euler characteristic as an index

Let M be a Riemannian closed (i.e. compact and boundaryless) manifold. As a consequence of the theory of elliptic operators on a compact manifold, for any $p \in \mathbb{N}$, the cohomology group $H^p(M)$ is a finite dimensional space and its dimension coincides with that of the space of harmonic p-forms $\mathcal{H}^p(M) := Ker\left(\Delta_{|_{\Omega^p(M)}}\right)$. Since $\mathcal{H}^p(M) \simeq H^p_{\text{sing}}(M) \simeq H_p^{\text{simpl}}(M)$ where $H^p_{\text{sing}}(M)$ is the singular cohomology group and $H_p^{\text{simpl}}(M)$ is the simplicial homology group, these have same dimension called the p-th *Betti number* and often denoted by

$$\beta_p(M) := \dim \mathcal{H}^p(M) = \dim H^p_{sing}(M) = \dim H_p^{simpl}(M).$$

The Euler characteristic is the alternating sum of the Betti numbers:

$$\chi(M) = \sum_{p=0}^n (-1)^p \beta_p(M).$$

Example 6 $\chi(S^n) = 1 + (-1)^n$ *so that* S^n *has Euler characteristic equal to* 2 *if* n *is even and* 0 *otherwise.*

The index of the Dirac operator $D_P^+ := (1 - P)(d + d^*)$ where P is the parity operator which is 1 on even forms and -1 on odd forms, can be expressed in terms of the Euler characteristic:

Lemma 9

$$\mathrm{ind}(D_P^+) = \chi(M).$$

Proof As before we set $D = d + d^*$.

$$\begin{aligned}
\mathrm{ind}(D_P^+) &= \mathrm{dimKer}(D_P^+) - \mathrm{dimKer}(D_P^-) \\
&= \dim\left(\mathrm{Ker}((1-P)D)\right) - \dim\left(\mathrm{Ker}((1+P)D)\right) \\
&= \sum_p \dim\left(\mathrm{Ker}(D_{|_{\Omega^{2p}(M)}})\right) - \sum_p \dim\left(\mathrm{Ker}(D_{|_{\Omega^{2p+1}(M)}})\right) \\
&= \sum_{p=0}^{n} (-1)^p \dim\left(\mathrm{Ker}(D_{|_{\Omega^{p}(M)}})\right) \\
&= \sum_{p=0}^{n} (-1)^p \dim \mathcal{H}^p(M) \\
&= \sum_{p=0}^{n} (-1)^p \beta_p \\
&= \chi(M).
\end{aligned}$$

□

The Atiyah-Singer index theorem (which we do not prove here) gives a local expression of the index:

Theorem 1

$$\mathrm{ind}(D_P^+) = \chi(M) = \frac{1}{\sqrt{2\pi}^n}\int_M e(\nabla^{TM})$$

where ∇^{TM} *is the Levi-Civita connection.*

There is another possible interpretation of the Euler characteristic as the index of the zero section in the tangent bundle TM. Let $f : M \to W$ be a smooth map from a closed oriented smooth m-dimensional manifold M to an oriented manifold W of dimension $m + n$ such that f is transverse to another closed oriented smooth n-submanifold N of M. A point $x \in f^{-1}(N)$ has positive or negative type according to whether the composition:

$$T_x M \to W_{f(x)} \to W_{f(x)}/N_{f(x)}$$

preserves or reverses orientation, where the first map is the tangent map to f. Accordingly we set $i_x(f, N) = 1$ or $i_x(f, N) = -1$. The *intersection number* of (f, N) is the integer:

$$i(f, N) := \sum_{x \in f^{-1}(N)} i_x(f, N).$$

It is finite and invariant under homotopies of the map f.

Now if $s_0 : M \to TM$ is the zero section of the tangent bundle of M, we have:

$$\xi(M) = i(s_0, M).$$

As a consequence, since any section s of the tangent bundle is homotopic to the zero section by the map $(t, x) \mapsto ts(x)$, if the tangent bundle to M has a section which is nowhere zero then $\xi(M) = 0$. If $\xi(M) \neq 0$, every vector field on M vanishes somewhere and the corresponding "hairy" manifold "cannot be combed". When n is even, this is the case of the n-dimensional sphere S^n which has Euler characteristic $\chi(S^n) = 2$ so that a hairy even dimensional ball cannot be combed.

9.4 The signature as an index

Let M be an even dimensional closed manifold with $\dim M = n = 2k$. The map

$$\begin{array}{rcl} \Omega^k(M) \times \Omega^k(M) & \to & \mathbb{R} \\ (\alpha, \beta) & \mapsto & \sigma(\alpha, \beta) := \int_M \alpha \wedge \beta \end{array}$$

yields a bilinear form. Since $\int_M \alpha \wedge \beta = (-1)^k \int_M \beta \wedge \alpha$, it is symmetric whenever $k = 2l$ is even. We now work in this setting so that $n = 4l$.

Lemma 10 *In dimension $n = 4l$, the chirality operator Γ and the Hodge star operator $\star$ coincide on $k = 2l$-forms:*

$$\Gamma_{|_{\Omega^k(M)}} = \star_{|_{\Omega^k(M)}}.$$

Proof Setting $p = 2l$ and $n = 4l$ in formula (7):

$$\Gamma = (-1)^{pn + \frac{p(p-1)}{2}} i^{k(n)} \star,$$

yields the the result of the proposition.

Corollary 3 *If $n = 2k = 4l, l \in \mathbb{N}$ the space*

$$\Omega^{k,sd}(M) := \{\alpha \in \Omega^k(M), \star\alpha = \alpha\}$$

of self-dual forms coincides with $\Omega^{k,+}(M) = \mathrm{Ker}(\Gamma - I) \cap \Omega^k(M)$ and the space

$$\Omega^{k,asd}(M) := \{\alpha \in \Omega^k(M), \star\alpha = -\alpha\}$$

of antiself-dual forms coincides with $\Omega^{k,-}(M) = \mathrm{Ker}(\Gamma + I) \cap \Omega^k(M)$.

Since the space of harmonic k-forms is finite dimensional, so are the spaces $\mathcal{H}^{k,sd}(M)$ of *self-dual harmonic k-forms* and $\mathcal{H}^{k,asd}(M)$ of *anti self-dual harmonic k-forms* with dimensions denoted by $\beta_k^+(M) := \dim\mathcal{H}^{k,sd}(M)$ and $\beta_k^-(M) := \mathcal{H}^{k,asd}(M)$ respectively. They coincide with the k-th de Rham cohomology spaces of self-dual, respectively anti-self dual forms.

Proposition 9 *In dimension $n = 2k = 4l$, σ induces a non degenerate symmetric bilinear form on $H^k(M)$. Its signature is given by*

$$\sigma(M) := \mathrm{sign}(\sigma) = \dim\mathcal{H}^{k,sd}(M) - \dim\mathcal{H}^{k,asd}(M) = \beta_k^+(M) - \beta_k^-(M)$$

where $\mathcal{H}^{k,sd}(M)$ is the (finite dimensional) space of self-dual harmonic k-forms and $\mathcal{H}^{k,asd}(M)$ is the (finite dimensional) space of antiself-dual harmonic k-forms.

Proof Given two closed forms α and β, $\sigma(\alpha, \beta)$ only depends on the cohomology class of α and β; indeed

$$\begin{aligned} \int_M (\alpha + d\gamma) \wedge \beta &= \int_M \alpha \wedge \beta + \int_M d\gamma \wedge \beta \\ &= \int_M \alpha \wedge \beta + \int_M d(\gamma \wedge \beta) - \int_M \gamma \wedge d\beta \\ &= \int_M \alpha \wedge \beta \end{aligned}$$

where we have used Stoke's theorem to set the middle integral to zero and the fact that β is closed to set the last integral to zero.

The form σ is non degenerate; indeed, let us assume that $\int_M \alpha \wedge \beta = 0$ for any closed k-form β and let us show that $\alpha = 0$. Pick a harmonic k-form α as representative of a cohomology class in $H^k(M)$. Then

$$(d + d^*)\alpha = 0 \Rightarrow d\alpha = d^*\alpha = 0 \Rightarrow d(\star\alpha) = 0.$$

Hence $\star\alpha$ is also a closed k-form and we can take $\beta = \star\alpha$. This yields $\alpha \wedge \star\alpha = \|\alpha\|^2 = 0$ so that $\alpha = 0$ which ends the proof of the non degeneracy.

To show that σ is diagonal on $H^{k,sd}(M) \oplus H^{k,asd}(M)$, let us compute $\sigma(\alpha, \beta)$ for various types of forms α and β. If $\alpha \in \mathcal{H}^{k,sd}(M)$, $\beta \in \mathcal{H}^{k,sd}(M)$

$$\sigma(\alpha, \beta) = \int_M \alpha \wedge \beta = \int_M \alpha \wedge \star\beta = \langle \alpha, \beta \rangle,$$

if $\alpha \in \mathcal{H}^{k,asd}(M)$, $\beta \in \mathcal{H}^{k,asd}(M)$

$$\sigma(\alpha, \beta) = \int_M \alpha \wedge \beta = \int_M \alpha \wedge \star\beta = -\langle \alpha, \beta \rangle$$

and if $\alpha \in \mathcal{H}^{k,sd}(M)$, $\beta \in \mathcal{H}^{k,asd}(M)$

$$\sigma(\alpha, \beta) = \int_M \star\alpha \wedge \beta = (-1)^k \int_M \beta \wedge \star\alpha = \langle \alpha, \beta \rangle = \int_M \star\alpha \wedge \star\beta = -\sigma(\alpha, \beta) = 0.$$

As a consequence, σ is diagonal on $\mathcal{H}^{k,sd}(M) \oplus \mathcal{H}^{k,asd}(M)$ with eigenvalues $+1, -1$ with multiplicity b_k^+ and b_k^-. It therefore has signature

$$\mathrm{sign}(\sigma) = \mathrm{dim}\mathcal{H}^{k,sd}(M) - \mathrm{dim}\mathcal{H}^{k,asd}(M) = \beta_k^+(M) - \beta_k^-(M).$$

□

Lemma 11 *For a manifold M of dimension $n = 4l$,*

$$\chi(M) = b_{2l} \mod 2.$$

Proof Setting $k = 2l$ we have (dropping the M for the moment)

$$\begin{aligned}
\chi(M) &= \sum_{p=0}^{n} (-1)^p \beta_p \\
&= (-1)^k b_k + \sum_{0<p<k}^{n} (-1)^p \beta_p + \sum_{0<p<k} (-1)^p \beta_{n-p} \\
&= (-1)^k \beta_k + 2 \sum_{0<p<k}^{n} (-1)^p \beta_p \\
&= \beta_k \mod 2
\end{aligned}$$

where we have used Poincaré duality $H^{n-p}(M) \simeq H^p(M)$ to identify $\beta_p(M)$ and $\beta_{n-p}(M)$.

Proposition 10

$$\sigma(M) = \chi(M) \quad \mathrm{mod}2.$$

Proof

$$\begin{aligned}
\sigma(M) &= \beta_k^+(M) - \beta_k^-(M) \\
&= \beta_k^+(M) + \beta_k^-(M) \quad \mathrm{mod}2 \\
&= \beta_k(M) \quad \mathrm{mod}2 \\
&= \chi(M) \quad \mathrm{mod}2.
\end{aligned}$$

□

The index of the signature operator

$$D_\Gamma^+ = (d + d^*)_{|_{\mathrm{Ker}(\Gamma - I)}}$$

coincides with the signature of M. □

Lemma 12

$$\mathrm{ind}(D_\Gamma^+) = \sigma(M).$$

Proof We first observe that if $\alpha_i, i = 1, \cdots, \beta_j(M)$ is an orthonormal basis of $\mathcal{H}^j(M)$ with $j < k$ then $\alpha_i^+ = \alpha_i + \star\alpha_i, \alpha_i^- = \alpha_i - \star\alpha_i, \quad i = 1, \cdots, \beta_j(M)$ yield an orthonormal basis of $\mathcal{H}^j \oplus \mathcal{H}^{n-j}$, where $\star$ is the Hodge star operator. Since the α_i^+ are self-dual and the α_i^- are anti self-dual, they yield an orthonormal basis respectively of $\mathcal{H}^{j,sd}(M) \oplus \mathcal{H}^{n-j,sd}(M)$ and $\mathcal{H}^{j,asd}(M) \oplus \mathcal{H}^{n-j,asd}(M)$. As a consequence,

$$\begin{aligned}
\beta_j^+(M) + \beta_{n-j}^+(M) &= \dim\left(\mathcal{H}^{j,sd}(M) \oplus \mathcal{H}^{n-j,sd}(M)\right) \\
&= \dim\left(\mathcal{H}^{j,asd}(M) \oplus \mathcal{H}^{n-j,asd}(M)\right) = \beta_j^-(M) + \beta_{n-j}^-(M).
\end{aligned}$$

Hence we have,

$$\begin{aligned}
\mathrm{ind}(D_\Gamma^+) &= \mathrm{dimKer}(D_\Gamma^+) - \mathrm{dimKer}(D_\Gamma^-) \\
&= \dim\Big(\mathrm{Ker}(D_{|_{\Omega^{sd}(M)}})\Big) - \dim\Big(\mathrm{Ker}(D_{|_{\Omega^{asd}(M)}})\Big) \\
&= \sum_{j=0}^{n} \mathrm{dimKer}\mathcal{H}^{j,sd}(M) - \sum_{j=0}^{n} \mathrm{dimKer}\mathcal{H}^{j,asd}(M) \\
&= \sum_{j=0}^{n} \beta_j^+(M) - \sum_{j=0}^{n} \beta_j^-(M) \\
&= \sum_{j=0}^{k-1} \left(\beta_j^+(M) + \beta_{n-j}^+(M)\right) - \sum_{j=0}^{k-1} \left(\beta_j^-(M) + \beta_{n-j}^-(M)\right) + \beta_k^+(M) - \beta_k^-(M) \\
&= \beta_k^+(M) - \beta_k^-(M) \\
&= \sigma(M).
\end{aligned}$$

□

The Atiyah-Singer index theorem (which we do not prove here) gives a local expression of the index as an integral of the $\hat{L}$ genus:

Theorem 2 *Let M be an oriented Riemannian closed $n = 4l$ dimensional manifold.*

$$\mathrm{ind}(D_\Gamma^+) = \sigma(M) = \frac{(-1)^l}{\pi^{2l}} \int_M L(\nabla^{TM})$$

where as before, ∇^{TM} is the Levi-Civita connection.

On a 4-dimensional manifold i.e $l = 1$, using the Taylor expansion of the L-genus, $L(\nabla^{TM}) = 1 + \frac{1}{24}\mathrm{tr}(\Omega^{TM} \wedge \Omega^{TM}) + \cdots$, we find that

$$\sigma(M) = -\frac{1}{24\pi^2} \int_M \mathrm{tr}(\Omega^{TM} \wedge \Omega^{TM}).$$

References

[APS I] M. Atiyah, V. Patodi, I.M. Singer *Spectral asymmetry and Riemannian Geometry I*, Math.Proc.Camb.Phil.Soc. **77** (1975) 43–69

[APS II] M. Atiyah, V. Patodi, I..M. Singer *Spectral asymmetry and Riemannian Geometry II*, Math.Proc.Camb.Phil.Soc. **78** (1975) 405–432

[APS III] M. Atiyah, V. Patodi, I..M. Singer *Spectral asymmetry and Riemannian Geometry III*, Math.Proc.Camb.Phil.Soc. **79** (1976) 71–99

[AS] M.Atiyah, I.M. Singer, *Dirac operators coupled to vector potentials*, Proc.Nath.Acad.Sci.USA, **81** (1984) 2597–2600

[BGV] N. Berline, E. Getzler, M. Vergne, **Heat kernels and Dirac operators**, Springer-Verlag, 1992

[F] T. Friedrich, **Dirac-Operatoren in der Riemanschen Geometrie**, Vieweg, Braunschweig-Wiesbaden, 1997

[GHL] S. Gallot, D.Hulin, J. Lafontaine, **Riemannian geometry**, Springer Verlag 1980, 2nd ed. 2002

[G] P.Gilkey, **Invariance theory, the heat equation and the Atiyah-Singer index theorem**, Studies in Advanced Mathematics 1995

[Hu] S.T. Hu, *Differentiable Manifolds*, Holt, Rinehart and Winston 1969

[Hus] D. Husemoller, **Fibre bundles**, McGraw-Hill Series 1966; 2nd edition, GTM vol. **20**, Springer Verlag 1975

[J] J. Jost, **Riemannian geometry and geometric analysis**, Springer Verlag, Berlin 1995; 2nd ed. 1998; 3rd ed. 2002

[K] W. Klingenberg, **Riemannian geometry**, de Gruyter Studies in Mathematics 1982

[KN] S. Kobayashi, N. Nomizu, **Foundations of differential geometry I, II**, Interscience, John Wiley 1963, 2nd ed. 1969

[L] S. Lang, **Differential and Riemannian Manifolds**, Springer 1995 (3rd edition of "Differential Manifolds", originally published by Addison-Wesley 1962)

[LM] B. Lawson and M. Michelson, **Spin geometry**, Princeton University Press, Princeton 1989

[MS] J. Milnor, J. Stasheff, **Characteristic classes**, Princeton University Press 1974

[MT] I. Masden, J. Tornehave, **From calculus to cohomology**, Cambridge University Press 1997

[M] Y. Matsushima, **Differentiable manifolds**, Pure and Applied Mathematics 1972

[Na] M. Nakahara, **Geometry, Topology and Physics**, Adam Hilger, 1990

[Ro] S. Rosenberg, **The Laplacian on a Riemannian manifold**, London Mathematical Society, Student Texts **31** 1997