CIMPA Scool in

Recent Topics in Geometric Analysis

Institute for Studies in Theoretical Physics and Mathematics

TEHRAN, IRAN, May 2006

Scientific Committee: Gilles Carron, Bruno Colbois, Thierry Coulhon, Ahmad El Soufi, Régis Monneau, Frank Pacard, Alireza Ranjbar-Motlagh, Tristan Riviére, Mehrdad Shahshahani.

Organizing Committee: Mehrdad Shahshahani (IPM-Tehran), Ahmad El Soufi (Tours, France), Alireza Ranjbar-Motlagh (Sharif University, Tehran, Iran) .

Lecturers:

- Gilles Carron, Professor, Université de Nantes, France
- Bruno Colbois, Professor, Université de Neuchâtel, Switzerland
- Thierry Coulhon, Professor, Université de Cergy-Pontoise, France
- Ahmad El Soufi, Professor, Université de Tours, France
- Régis Monneau, Professor, Ecole Nationale des Ponts et Chaussées, Paris, France
- Frank Pacard, Professor, Université de Paris XII, France
- Alireza Ranjbar-Motlagh, Professor, Sharif University, Iran
- Tristan Riviére, Professor, ETH-Zürich, Switzerland
- Mehrdad Shahshahani, Professor, IPM, Iran.

PROGRAM OF COURSES

1. **Cohomologie et formes harmoniques des carrés sommables sur les variétés non compacte:** (*Gilles Carron*)

   Les formes harmoniques de carres intégrable sur les variétés riemanniennes complétes non-compactes dépendent la fois de la géométrie et de la topologie de la variété. On veut donner des outils assez simple (de nature géométrique ou topologique) qui permettent de décrire ces espaces.

   Plan du cours:

   - Cohomologie de carres intégrable versus formes harmoniques de carres intégrable, théorème de Hodge-deRham.
   - Quelques proprietes sur les varietes non compactes: La finitude des espaces de formes harmoniques de carres integrable ne depend que de linfini. Quelques exemples de calculs en utilisant linvariance conforme des formes de degré égale la dimension moitié : cas des surfaces non compactes et des métriques ALF (asymptote la metrique TUB-NAUT).
   - Nombres de bouts dune variété non-compacte et formes harmoniques de carres intégrable de degré 1. Exemples dapplications (l'homologie en co-dimension 1 des hypersurfaces minimales stables d'espéce euclidean dimension n est trivial si $n > 3$.
   - Topologie et formes harmoniques de carres intégrable: limage de la cohomologie support compact dans la cohomologie sinjecte dans lespace des formes harmoniques de carres intégrable. Conditions pour avoir égalité et lien avec la parabolicité. On donnera une preuve unifié de certains résultats de, suivant la méthode de Carpar.

2. **The spectrum of the Hodge-De Rham Laplacian:** (*Bruno Colbois*)

   We study the relationship between the geometry and the topology of a compact Riemannian manifold (M, g) through the spectrum of the Laplacian acting on p-forms, which is known to be discrete, with a particular interest in the first non-zero eigenvalue. Classically, as regards functions, the goal is to estimate the spectrum with the Riemannian invariants of the manifold, in particular curvature, diameter, and volume. However, for p-forms, the importance of the topology of the manifold appears clearly (recall that the multiplicity of the eigenvalue 0 corresponds to the p^{th} Betti number of M and is a topological invariant), in particular via the collapsing. The goal of these lectures is to give an introduction to this subject and also to give some recent developments and open questions.

Course Plan

- Presentation of the problem. The case of function on domains and on compact Riemannian manifolds, with some classical results.
- Examples for the p-form spectrum: the case of collapsing manifolds and related topics.
- How to get some lower bound on the p-form spectrum. Some recent developments and open questions.

3. **Heat kernel and Riesz tranform:** (*Thierry Coulhon*)

   Let M be a complete non-compact Riemannian manifold, μ the Riemannian measure, r the Riemannian gradient, and ∇ the (positive) Laplace-Beltrami operator on M. Denote by $|.|$ the length in the tangent space, and by $||.||_p$ the norm in $L^p(M;\mu)$, $1 \leq p \leq \infty$. It was asked in by Strichartz in 1983 for which complete non-compact Riemannian manifolds M and which $p \in (1, \infty)$ one has

$$C_p^{-1}||\Delta^{\frac{1}{2}} f||_p \leq ||\nabla f||_p \leq C_p||\Delta^{\frac{1}{2}} f||_p. \tag{1}$$

   For $p = 2$, on any complete Riemannian manifold, one has the equality

$$||\nabla f||_2 = ||\Delta^{\frac{1}{2}} f||_2$$

   and this may even be used to define the Laplace-Beltrami operator. For $p \neq 2$, already in the Euclidean space, the above equivalence of semi-norms is by no means a trivial matter, and a lot of work has been devoted to its proof for several classes of manifolds. In the last few years, it has appeared that, at least in the class of manifolds with the doubling property, the validity of (1) is strongly related with estimates of the heat kernel and its gradient. The aim of this course will be to explain these results.

4. **Spectral Geometry:** (*Ahmad El Soufi*)

   The first aim of this course is to familiarize the audience with the spectrum of the Laplace operator on Riemannian manifolds, especially in order to supply a necessary background for the other courses of the school (in particular, the courses of Carron and Colbois). Hence, the first part of the course will consists on:

   - Basic facts about the spectrum of the Laplacian including discretness and finite degeneracy in the compact case.
   - Survey of classical results (Lichnerowicz-Obata, Cheng, Cheeger, Buser) on relationships between eigenvalues and other geometric invariants (curvature, diameter, injectivity radius, isoperimetric constants etc.)

- Discussion of isoperimetric problems related to the eigenvalues of Laplacian : Bergers work and results of Hersch, Li and Yau, Yang and Yau, El Soufi and Ilias, ...

In the last part of the course, we will discuss some recent results and open problems about extremal geometries for the eigenvalues of the Laplacian.

5. **Geometric evolutions via partial differential equations:** (*Régis Monneau*)

   We will give an introduction to continuous and discontinuous viscosity solutions for first and second order PDE. In the particular case of geometric partial differential equations, the level set of a function describes the geometric dynamics of a hypersurface. We will also present a recent and direct approach to geometric evolutions of hypersurfaces, without using the level sets approach. We will illustrate these geometric dynamics both on classical dynamics like the mean curvature motion, and also on new dynamics like non-local motion for dislocation dynamics.

6. **The singular Yamabe problem:** (*F. Pacard*)

   Geometrically the problem is to find, on subdomains of the n-dimensional sphere ($n > 2$), complete metrics which are conformal to the standard metric and have constant scalar curvature. The study of this problem has been initiated by R. Schoen and S.T. Yau since it plays a crucial role in the classification of locally conformally flat, complete manifolds with constant scalar curvature. This geometric problem reduces to the study of singular solutions of semilinear elliptic equations. In this course we explain how to solve the singular Yamabe problem in various situations. This will be the opportunity to introduce tools of the analysis of elliptic operator on compact manifolds which have been developed by Mazzeo and Melrose.

7. **Isoperimetric Inequalities, Sobolev Constants and Heat Kernel:** (*Alireza Ranjbar-Motlagh*)

   We first introduce the isoperimetric inequalities and their relations with the Sobolev and Poincaré type inequalities on Riemannian manifolds. We discuss the geometric and analytic implications of the isoperimetric and Faber-Krahn type inequalities, their relation to bounds for heat kernels and their geometric and analytical implications. Finally, we introduce the Poincaré and Sobolev inequalities in the framework of abstract spaces.

8. **Concentration compactness in high dimensions and geometric applications:** (*Tristan Rivière*)

   - We will recall the fundamental result of Sachs and Uhlenbeck regarding the loss of compactness of sequences of harmonic maps from a surface into a manifold.

We will explain how the result extends to a large class of conformally invariant geometric PDE. We will discuss the problem of extending the description of the loss of compactness of these PDE in dimension larger than the conformal one and explain the geometric motivation behind. To that purpose, we will study the particular case of Yang-Mills equations in dimension larger than 4 and explain how Sachs-Uhlenbeck's result can be generalized to this high dimension situation. We will finish the course by describing the still existing difficulties of giving a satisfactory description of the Moduli Space of Yang-Mills in high dimension and look at particular cases such as $SU(4)$ instantons on Calabi-Yau 4-folds.

9. **Analysis on Symmetric Spaces:** (*Mehrdad Shahshahani*)

- Symmetric Spaces and invariant differential operators.
- The notion of Fourier transform on symmetric spaces.
- Theorems of Plancherel and Paley-Wiener.
- Applications to analysis on symmetric spaces.
- The structure of arithmetically defined locally symmetric spaces.
- The decomposition of the spectrum of a locally symmetric space.
- Applications of spectral decomposition to automorphic forms and number theory.