Finite Quasi-Frobenius bimodules

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Interest to linear codes over finite ring (f.r.) and modules was renew last time and based on the discovery by author (Nechaev 1982-89 [19, 20]) the effect of linear representation over ring $\mathbb{Z}_4$ of some nonlinear codes over field (binary Kerdock codes) and on later investigations and generalizations of this result in works Hammons, Kumar, Calderbank, Sloane, Sole (1994 [11]) and Nechaev, Kuzmin (1993-96 [21, 22, 23]).

The foundations of classical algebraic coding theory over finite fields are characterize first of all by notions and results like **dual code**, **MacWilliams identity** and **extension theorem**.

The attempts to generalize this theory on linear codes over modules advance in different directions:

– linear codes over finite modules with commutative coefficient ring

(Kuzmin, Kurakin, Markov, Mikhalev, Nechaev [31] — [34]);

– linear codes over finite Frobenius rings and chain rings

(Greferath, Honold, Landgev, Lopez, Smidt, Wood. [43, 44, 9, 15]);

–weights on finite rings and modules

(Heise, Constantinescu, Honold, Nechaev, Greferath [5, 13, 16, 9]);

–general theory of linear codes over finite modules

(Greferath, Nechaev and Wisbauer [10]).

These results educe an exceptional role of **quasi-Frobenius modules** in all aspects of the theory.

BASIC EXAMPLE

Let R be a finite (in general noncommutative) ring with identity $1 = 1_R$ (below: f.r.). Let ${}_RM$ be a finite faithful module. Any submodule $\mathcal{K} \leq {}_RM^n$ is called a **(left) linear n-code over ${}_RM$**.

For the development of such codes theory it is necessary first to introduce some correctly conceptions of a parity-check matrix and a dual to $\mathcal{K}$ code $\mathcal{K}^o$, in particular so, that $\mathcal{K}^{oo} = \mathcal{K}$.

For example, if $\mathcal{K} < {}_RR^n$, we can try to define $\mathcal{K}^o$ in a usual way as a right linear code

$$\mathcal{K}^o = \{\beta = (b_1, ..., b_n) \in R^n : \forall \alpha \in \mathcal{K} \;\; \alpha\beta = 0\} \leq R^n_R,$$

where $\alpha\beta = a_1b_1 + ... + a_nb_n$. Then dual to $\mathcal{K}^0$ left linear code satisfies the relation $\mathcal{K}^{oo} \supseteq \mathcal{K}$, but

$$(\mathcal{K}^{oo} = \mathcal{K} \;\text{ for all } \mathcal{K} \leq {}_RR^n) \iff (R \;\text{ is a quasi-Frobenius ring}).$$

So linear codes over a QF-ring R can be studied without codes over modules, as it was made in earlier works about linear codes over residual rings and in works of Greferath, Honold, Landgev, Lopez and Wood [43, 44, 15].

If R in not a QF-ring, then right construction of a dual to $\mathcal{K}$ code is a linear code not over R but over the QF-module corresponds to R.

1 Characterizations of quasi-Frobenius bimodules

Suppose A and B are finite, not necessarily commutative rings with identities and ${}_AM_B$ be a left and right faithful (A,B)-bimodule:

$$\forall\ a\in A,\ b\in B,\ \alpha\in M:\quad (a\alpha)b=a(\alpha b);$$

$$(aM=0)\Rightarrow(a=0);\quad (Mb=0)\Rightarrow(b=0).$$

Let $\mathrm{End}(M_B)$ and $\mathrm{End}({}_AM)$ be endomorphism rings, where the elements of $\mathrm{End}({}_AM)$ act on elements of M from the right, and elements of $\mathrm{End}(M_B)$ act on elements of M from the left. Then agreeing with some natural identification we can consider B as a subring of the ring $\mathrm{End}({}_AM)$ and can consider A as a subring of the ring $\mathrm{End}(M_B)$. Really we can identify $b\in B$ with a map

$b: M\to M$ by the rule $\forall\alpha\in M\ \ \alpha\to\alpha b$.

Then $b\in\mathrm{End}({}_AM)$. Symmetrically we identify $a\in A$ with a map

$a: M\to M$ by the rule $\forall\alpha\in M\ \ \alpha\to a\alpha$.

Then $a\in\mathrm{End}(M_B)$.

1.1 Definition and properties of QF-bimodules

A bimodule ${}_AM_B$ is called **quasi-Frobenius** (QF-bimodule) (Azumaya [1]), or **duality context** (Faith [8]), if for every maximal left ideal $I \lhd A$ its (right) annihilator in M:

$$\rho_M(I) = \{\beta \in M \mid I\beta = 0\}$$

is zero or an irreducible B-module, and for every maximal right ideal $J \lhd B$ its (left) annihilator in M:

$$\lambda_M(J) = \{\alpha \in M \mid \alpha J = 0\}$$

is zero or an irreducible A-module. A ring R is called **quasi-Frobenius** if the natural bimodule ${}_RR_R$ is OF.

Example 1. Any finite PIR R in particular $R = \mathbb{Z}_m$; is a QF-ring. Really if $R = \mathbb{Z}_m$ and $I <_{max} {}_RR$ then $I = pR$, where $p \mid m$ and p is a prime. Therefore $\rho_R(I) = dR$, where $d = m/p$, and dR is an irreducible (minimal) ideal of R of the cardinality p.

Example 2. Let $P = GF(q)$, G be a finite ring, then group ring PG is a QF-ring.

Example 3. Let $P = GF(q)$, $A = M_m(P)$, $B = M_n(P)$, then ${}_AM_B$, where $M = M_{m,n}(P)$ is a QF-bimodule.

Using [1, Prop. 3, Th. 6](Azumaya) and [8, Th 23.25](Faith) we have

Theorem 1.1. *For the finite rings* A, B *and a faithful bimodule* ${}_AM_B$ *the following conditions are equivalent.*

- **(a)** ${}_AM_B$ *is a (finite) QF-bimodule.*
- **(b)** $A = \mathrm{End}(M_B)$, $B = \mathrm{End}({}_AM)$ *and for every submodules* $L \leq {}_AM$ *and* $N \leq M_B$ *there hold*

$$\lambda_M(\rho_B(L)) = L, \quad \rho_M(\lambda_A(N)) = N. \tag{1.1}$$

If ${}_AM_B$ *is a QF-bimodule then also for every left ideal* $I \leq {}_AA$ *and right ideal* $J \leq B_B$ *there hold*

$$\lambda_A(\rho_M(I)) = I, \quad \rho_B(\lambda_M(J)) = J. \tag{1.2}$$

1.2 Socle characterisation of QF-bimodule

Let us remind that **(nil-)radical** of a finite ring R with identity is a sum $\mathfrak{N} = \mathfrak{N}(R)$ of all its left nilpotent ideals, and it is the unique maximal two-sided nilpotent ideal contained all on-sided nilpotent ideals. For a finite ring R with identity $\mathfrak{N}(R) = \mathcal{J}(R)$ — **Jacobson radical** = intersection of all right maximal ideals of R.

The **socle** of ${}_AM$ is the notion dual to the notion of Jacobson radical, it is the sum $\mathfrak{S}({}_AM)$ of all left minimal (irreductible) submodles of ${}_AM$. It is also a right annihilator of $\mathfrak{N} = \mathfrak{N}(A)$ in M:

$$\mathfrak{S}({}_AM) = \rho_M(\mathfrak{N}) = \{\alpha \in M : \mathfrak{N}\alpha = 0\}.$$

In fact A-module $\mathfrak{S}({}_AM)$ is a left module over the top-factor $\overline{A} = A/\mathfrak{N}$ where multiplication of $\alpha \in \mathfrak{S}({}_AM)$ by $\overline{a} = a + \mathfrak{N} \in \overline{A}$ is defined as $\overline{a}\alpha = a\alpha$.

We have the following useful addition to the Theorem 1.1

Theorem 1.2. (Nechaev, 2000 [35]) *A faithful bimodule* ${}_AM_B$ *is QF iff*

(**c**) $\mathfrak{S}({}_AM) = \mathfrak{S}(M_B) = \mathfrak{S}$ *and* ${}_{\overline{A}}\mathfrak{S}_{\overline{B}}$ *is a QF-bimodule.*

Example 4. The ring R of all upper-triangle 2×2-matrices over a field P is not a QF-ring. Really

$\mathfrak{N}(R)$ is a subset of all matrices from R with zero diagonal;

$\mathfrak{S}({}_RR) = \lambda_R(\mathfrak{N}(R))$ is a subset of all matrices with zero first column;

$\mathfrak{S}(R_R) = \rho_R(\mathfrak{N}(R))$ is a subset of all matrices with zero second row.

So $\mathfrak{S}({}_RR) \neq \mathfrak{S}(R_R)$.

In connection with the points (b) of Theorem 1.1 we shall call a module ${}_AM$ a **QF-module** if the natural bimodule ${}_AM_B$, where $B = \mathrm{End}({}_AM)$ is a QF-bimodule. It is well known (cf. [8]) that for every commutative finite ring R with identity there exists unique up to isomorphism QF-module ${}_RQ$, hereby $\mathrm{End}({}_RQ) = R$ and corresponding QF-bimodule is ${}_RQ_R$. This fact has been the basis for the results about properties of linear codes over finite modules with commutative coefficient ring (Kuzmin, Kurakin, Markov, Mikhalev, Nechaev [31, 32, 33, 34]).

Now we have natural **question**: does for a given finite not necessary commutative ring R exists a QF-module ${}_RM$?

Below we have the extension of the existence theorem to any finite ring with identity (Greferath, Nechaew, Wisbauer [10]).

2 EXISTENCE OF QF-MODULE.

2.1 Character Module.

Let ${}_AM$ be a finite module and $M^\flat = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ be the group of rational characters of the group $(M,+)$ with natural addition. Then

$$(M^\flat, +) \cong (M, +), \tag{2.1}$$

and there is a natural identification $M = M^{\flat\flat}$ by understanding the element $x \in M$ as a character of the group $(M^\flat, +)$, where the action is given by $x(\omega) := \omega(x)$ for all $\omega \in M^\flat$. For subgroups

$N \leq (M,+)$, $W \leq (M^\flat, +)$ we define their **map-annihilators**:

$$N^\perp := \{\omega \in M^\flat : \forall x \in N : \quad \omega(x) = 0\},$$

$$W^\perp := \{x \in M : \forall \omega \in W : \quad \omega(x) = 0\}.$$

Then $W^\perp \leq (M,+)$, $N^\perp \leq (M^\flat, +)$ and there are the equalities

$$N^{\perp\perp} = N, \quad W^{\perp\perp} = W. \tag{2.2}$$

Since M is a left A-module we can consider $M^\flat$ as a right A-module defining

$$\forall \ \omega \in M^\flat, \ a \in A, \ x \in M : \ \ (\omega a)(x) \ := \ \omega(ax) \tag{2.3}$$

Symmetrically, if M_B is a right B-module then $M^\flat$ is a left B-module with $(b\omega)(x) \ := \ \omega(xb)$ for all $\omega \in M^\flat, x \in M$ and $b \in B$.

Proposition 2.1. *Let $N \le (M.+)$ and $W \le (M^\flat, +)$. Then*

$$N \le {}_RM \Rightarrow N^\perp \le M^\flat_R, \qquad W \le M^\flat_R \Rightarrow W^\perp \le {}_RM.$$

2.2 Construction of QF-module.

Let now $M = R$ be a finite ring with identity. Then we have bimodules ${}_RR_R$ and ${}_RR^\flat_R$. and hence the map-annihilator of a left (right) ideal of R is a right (left) submodule of $R^\flat$. But even more is true.

Proposition 2.2. *Let $I \le {}_RR, \ J \le R_R, \ L \le {}_RR^\flat$ and $N \le R^\flat_R$ then*

$$\rho_{R^\flat}(I) = I^\perp, \quad \lambda_{R^\flat}(J) = J^\perp. \tag{2.4}$$

$$\rho_R(L) = L^\perp, \quad \lambda_R(N) = N^\perp. \tag{2.5}$$

Now we have the following existence Theorem for QF-modules.

Theorem 2.3. *For every finite ring R the module ${}_RR^\flat$ is a QF-module (i.e. ${}_RR^\flat_R$ is a QF-bimodule).*

2.3 Frobenius rings and bimodules

A f.r. R is called **quasi-Frobenius** or **QF-ring** if bimodule ${}_RR_R$ is QF. A QF-ring R is called **Frobenius** if

$$ {}_{\overline{R}}\overline{R} \cong {}_{\overline{R}}\mathfrak{S}(R), \quad \text{and} \quad \overline{R}_{\overline{R}} \cong \mathfrak{S}(R)_{\overline{R}}. $$

In the finite context this can be simplified.

Theorem 2.4. (Honold [17]) *A finite ring R with identity is a Frobenius ring iff $\mathfrak{S}({}_RR)$ is a left principal ideal.*

REMARK. This Theorem is interesting continuation of a series of

(Left) $\Rightarrow$ (Right) Theorems for f.r. :

1. If R is a f.r. and any two-sided ideal of R is **left** principal **then** R is a PIR (and any two sided ideal is **right** principal).
2. If R is a f.r. with identity and any two-sided idempotent ideal of R is **left** principal **then** R is a Wedderburn ring (and any idempotent ideal is **right** principal).
3. Theorem 2.4 (if $\mathfrak{S}({}_RR)$ is a **left** principal **then** $\mathfrak{S}({}_RR) = \mathfrak{S}(R_R)$ and it is a **right** principal ideal).

In analogy with the definition of a Frobenius ring we call a finite QF-bimodule ${}_AQ_B$ **Frobenius bimodule**, if

$$ {}_{\overline{A}}\overline{A} \cong {}_{\overline{A}}\mathfrak{S}(Q), \quad \text{and} \quad \overline{B}_{\overline{B}} \cong \mathfrak{S}(Q)_{\overline{B}}; \tag{2.6} $$

The condition (2.6) gives some hard restriction on rings A, B.

Proposition 5. (Nechaev 2004,[36].) *Let ${}_AQ_B$ be a Frobenius bimodule. Then $\overline{A} \cong \overline{B}$.*

Whether in this situation that $A \cong B$ — is an **open question**?

The following result prove in particular the existence of Frobenius bimodule for the case $A \cong B$. We know now that for every f.r. R there hold the relations

$$ \mathfrak{S}({}_RR^\flat) = \mathfrak{S}(R^\flat_R) = \mathfrak{S}(R^\flat). \tag{2.7} $$

Above that we have

Theorem 2.5. *For every f.r. R there exits an isomorphism of bimodules*

$$ {}_{\overline{R}}\overline{R}_{\overline{R}} \cong {}_{\overline{R}}\,\mathfrak{S}(R^\flat)_{\overline{R}}. \tag{2.8} $$

There exists a generator $\omega \in \mathfrak{S}(R^\flat)$ such that

$$ \mathfrak{S}(R^\flat) = \overline{R}\omega = \omega\overline{R}, \quad \overline{r}\omega = \omega\overline{r} \ \text{ for all } \overline{r} \in \overline{R}. $$

In particular ${}_RR^\flat_R$ is a Frobenius bimodule.

The full description of Frobenius $(R,R)-$ bimodules with a given finite coefficient ring R is the following. For any fixed $\theta \in Aut(R)$ we can define a structure of (R,R)-bimodule on the group $(R^\flat,+)$ by the conditions:

$$\forall\ a \in R,\ \omega \in R^\flat,\ x \in R: \quad (a\omega)(x) = \omega(xa), \ \ (\omega a)(x) = \omega(\theta(a)x).$$

We denote this bimodule by ${}_R R^\theta_R$. For $\theta = 1$ we have ${}_R R^1_R = {}_R R^\flat_R$.

Theorem 2.6. (Nechaev 2004,[36].) *For a faithful bimodule* ${}_R Q_R$ *the following conditions are equivalent:*

(a) ${}_R Q_R$ *is a Frobenius bimodule;*

(b) $\mathfrak{S}({}_R Q) = \mathfrak{S}(Q_R) = \mathfrak{S}$ *and* ${}_{\overline{R}}\mathfrak{S}_{\overline{R}}$ *is a Frobenius bimodule;*

(c) $\mathfrak{S}({}_R Q)$ *is a left cyclic* R*-module;*

(d) ${}_R Q \cong {}_R R^\flat$*;*

(e) ${}_R Q_R \cong {}_R R^\theta_R$ *for some* $\theta \in Aut(R)$*.*

The equivalency of p.p. (a) and (c) is generalization of the Honold's Theorem 2.4 (proved equivalency (a) and (c) only for the case $Q = R$).

Theorem 2.7. *Let* $\mathrm{Inn}(R)$ *be the group of inner authomorphisms of the ring* R, *then*

$$\forall\,\theta,\tau\in Aut(R)\ \ ({}_RR_R^{\theta}\cong{}_RR_R^{\tau})\Leftrightarrow(\theta\equiv\tau\pmod{\mathrm{Inn}(R)}).$$

The number of classes of isomorphic Frobenius bimodules ${}_RQ_R$ *equals to* $|Aut(R)/\mathrm{Inn}(R)|$.

2.4 Frobenius rings and symmetric rings

Recall that a f.r. R is called quasi-Frobenius (Frobenius) if so is bimodule ${}_RR_R$. Note that in general case if R is a noncommutative Frobenius ring we can not state that there exists isomorphism of Frobenius bimodules ${}_RR_R$ and ${}_RR^\flat_R$.

We call a character $\varepsilon \in R^\flat$ **left generating** or **left distinguished** if $R^\flat = R\varepsilon$. The last equality is equivalent to $\lambda_R(\varepsilon) = 0$ which means that the kernel $\operatorname{Ker}\varepsilon = \varepsilon^\perp$ of homomorphism $\varepsilon : R \to \mathbb{Q}/\mathbb{Z}$ does not contain any of nonzero left ideals. A character that is left and right generating is called a **generating character**.

Theorem 2.8. (Wood 1997) *For a f.r. R every left (or right) generating character is generating, and the following statements are equivalent:*

- **(a)** *R is a Frobenius ring.*
- **(b)** *R has a (left) generating character ε.*
- **(c)** *There exists isomorphism $\varphi : {}_RR \to {}_RR^\flat$.*
- **(d)** *There exists isomorphism $\psi : R_R \to R^\flat_R$.*

Again (Left) $\Rightarrow$ (Right) Theorem!

Under the condition (b) of Theorem isomorphisms from points (c,d) can be chosen in form

$$\varphi(a) = a\varepsilon, \quad \psi(a) = \varepsilon a. \tag{2.9}$$

For a QF-ring R there exist two QF-bimodules: ${}_RR_R$ and ${}_RR^\flat_R$. In light of Theorem 2.8 we can state that ${}_RR \cong {}_RR^\flat$ if and only if the ring R is Frobenius and in the last case also $R_R \cong R^\flat_R$. However, even in this case we can not state that bimodules ${}_RR_R$ and ${}_RR^\flat_R$ are isomorphic because isomorphisms (2.9) can be different. A finite ring R is called **symmetric**, if

$${}_RR_R \cong {}_RR^\flat_R. \tag{2.10}$$

Of course any symmetric ring is a Frobenius one (Theorem 2.8).

We have the following characterization of symmetric rings. Let

$$K(R) \ := \ {}_{\mathbb{Z}}\langle ab - ba \mid a, b \in R\rangle.$$

Theorem 2.9. *A f.r. R is symmetric iff it has a generating character $\varepsilon \in R^\flat$ such that $\varepsilon(K(R)) = 0$.*

Corollary 2.10. *If R is a symmetric ring then $K(R)$ does not contain any nonzero left or right ideals of R.*

The converse of the latter statement is an **open question**.

The class of finite symmetric rings is large enough.

Proposition 2.11. *The following f.r. with identity are symmetric:*

(a) *all finite commutative Frobenius rings (in part. all finite commutative PIR rings);*

(b) *all finite Frobenius rings* R *with* $\mathrm{Aut}(R) = \mathrm{Inn}(R)$ *;*

(c) *every ring-direct sum of symmetric rings;*

(d) *full matrix rings over symmetric rings;*

(e) *every finite group ring over a symmetric ring.*

Corollary 2.12. *Every finite semisimple ring is symmetric.*

Lastly note that there exist finite Frobenius nonsymmetric rings.

Example 6. Let P=GF(q) be a finite field with a nontrivial automorphism σ and let $P[x;\sigma]$ be an Ore polynomial ring with multiplication defined for $a \in P$ by $xa = \sigma(a)x$. Then $R = P[x;\sigma]/(x^2)$ is a finite local PIR, and hence Frobenius ring, consisting of elements $\alpha = a_0 + a_1 z, \quad a_0, a_1 \in P, \quad z = x + (x^2)$ [37]. The unique proper ideal of R is $\mathfrak{N}(R) = Rz = Pz$.

Consider the $K(R)$. For any $\alpha \in R$ and $\beta = b_0 + b_1 z \in R$ we have

$$\alpha\beta - \beta\alpha = (a_1(\sigma(b_0) - b_0) + b_1(\sigma(a_0) - a_0))z.$$

It is evident that the set of all such differences is $Pz = Rz$ and $K(R) = \mathfrak{N}(R)$ is a nonzero ideal. So R is not a symmetric ring.

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