Symmetrization methods and applications to PDE's

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Recent Developments in the Theory of Elliptic PDE
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Applications of isoperimetric inequality

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Pólya-Szegö inequality

Let $u \in W^{1,p}(\mathbb{R}^n)$ be a non-negative function with compact support, then:

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If the equality holds, is it true that $u = u^\sharp$, modulo translations?

Yes, if the following condition is assumed:

$$|\{0 < u^\sharp < \sup u\} \cap \{|Du^\sharp| = 0\}| = 0.$$

[J.E. Brothers - W.P. Ziemer, 1988]

First eigenvalue (Lord Rayleigh)

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Applications of isoperimetric inequality

Comparison results for elliptic equations

Consider the problems $(a_{ij}\xi_i\xi_j \geq |\xi|^2)$

$$\begin{cases} -(a_{ij}u_{x_i})_{x_j} = f & \text{in } E \\ u = 0 & \text{on } \partial E \end{cases} \qquad \begin{cases} -\Delta v = f^\sharp & \text{in } E^\sharp \\ v = 0 & \text{on } \partial E^\sharp \end{cases}$$

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then

$$u^\sharp(x) \leq v(x), \qquad \text{in } E^\sharp.$$

The equality case has been characterized.

[H.F. Weinberger, 1962], [V.G. Maz'ja, 1969], [G. Talenti, 1976], [A. Alvino - P.-L. Lions - G. Trombetti, 1986], [S. Kesavan, 1988], [V.F. - M.R. Posteraro, 1991]

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For the sake of simplicity we assume that the data are regular and, furthermore, $f \geq 0$, then $u \geq 0$.

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Main ingredients of the proof

- Integration on the levels of u
- Gauss theorem
- Co-area formula
- Isoperimetric inequality
- Rearrangements properties

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$$= \int_{u=t} a_{ij} u_{x_j} \frac{u_{x_i}}{|\nabla u|} \geq$$

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&= \int_{u=t} a_{ij} u_{x_j} \frac{u_{x_i}}{|\nabla u|} \geq && \text{(ellipticity)} \\
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&\geq \int_{u=t} |\nabla u| \geq && \text{(Cauchy-Schwarz)} \\
&\geq \frac{\left(\int_{u=t} 1 \right)^2}{\int_{u=t} \frac{1}{|\nabla u|}}
\end{aligned}$$

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$$\geq \frac{\left(n\omega_n^{1/n}\mu_u(t)^{1-1/n}\right)^2}{-\mu_u'(t)} \qquad \mu_u(t) = |\{u > t\}|$$

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Hardy-Littlewood gives

$$\int_{u>t} f \leq \int_0^{\mu_u(t)} f^*$$

then

$$\frac{n^2\omega_n^{2/n}\mu_u(t)^{2-2/n}}{-\mu_u'(t)} \leq \int_0^{\mu_u(t)} f^*$$

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$$u^\sharp(x) = u^*(\omega_n |x|^n) \le \frac{1}{n^2 \omega_n^{2/n}} \int_{\omega_n |x|^n}^{|E|} \frac{1}{r^{2-2/n}} \int_0^r f^* = v(x)$$

where $v(x)$ is the solution of the problem

$$\begin{cases} -\Delta v = f^\sharp & \text{in } E^\sharp \\ v = 0 & \text{on } \partial E^\sharp \end{cases}$$

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Analogous comparison results can be proven for several classes of partial differential equations.

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- Elliptic equations with lower-order terms

$$-(a_{ij}u_{x_i})_{x_j} - (b_i u)_{x_i} + c_i u_{x_i} + du = f$$

[A. Alvino - G. Trombetti, 1979], [G. Talenti, 1985], [V.F. - M.R. Posteraro, 1990], ...

Applications of isoperimetric inequality

Analogous comparison results can be proven for several classes of partial differential equations.

- Elliptic equations with lower-order terms
- Equations containing *p*-laplacian operator

$$-\Delta_p u \equiv -\text{div}(|Du|^{p-2}Du) = f$$

[G. Talenti, 1979], [C. Maderna - S. Salsa, 1987], [A. Alvino - P.-L. Lions - G. Trombetti, 1990], [M.F. Betta - V.F. - A. Mercaldo, 1994], ...

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- Equations containing the so-called pseudo-p-laplacian operator

$$\widetilde{\Delta}_p u \equiv (|u_{x_i}|^{p-2} u_{x_i})_{x_i} = \mathrm{div}(H(Du)^{p-1} H_{\xi_i}(Du)),$$

where $H(\xi) = (\sum_i |u_{x_i}|^p)^{1/p}$.

[A. Alvino - V.F. -P.-L. Lions - G. Trombetti, 1997], [M. Belloni - V.F. - B. Kawohl, 2003], ...

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- Hessian type equations like, for example, Monge-Ampère equation

$$Det(D^2u) = f.$$

[G. Talenti, 1981], [K. Tso, 1989], [N.S. Trudinger, 1997], ...

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- Hessian type equations like, for example, Monge-Ampère equation
- Problems with Neumann boundary conditions
- Parabolic equations

  [C. Bandle, 1976], [J. Mossino - R. Temam, 1981], [J.I. Díaz - J. Mossino, 1985], [J. Mossino - J.M. Rakotoson, 1986], [A. Alvino - P.-L. Lions - G. Trombetti, 1990], . . .

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Some consequences of comparison results

- Optimal information on the regularity of the solutions, both with respect to the summability and with respect to continuity properties. In particular, it is possible to successfully handle the cases where the data are in spaces with "limit" summability.

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- For some nonlinear equation the existence of a solution is related to some condition to be required on the norm of the data. In such cases the estimates obtained via symmetrization allow to establish optimal assumptions for the existence.

- In some cases it has been possible to use symmetrization methods and the isoperimetric property of the ball in order to prove symmetry properties for solutions to problems with symmetric data and for solutions to overdetermined problems.

Equations with natural growth in the gradient

Let us consider the following problem ($a_{ij}\xi_i\xi_j \geq |\xi|^2$).

$$\begin{cases} -(a_{ij}u_{x_i})_{x_j} = H(x, Du) & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

On the term on the right-hand side we make the following assumption

$$|H(x,\xi)| \leq |\xi|^2 + f(x), \qquad f \in L^\infty(\Omega).$$

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It is possible to obtain a comparison result between the bounded solution to the given problem and the bounded solution, if it exists, of the following symmetrized problem:

$$\begin{cases} -\Delta v = |Dv|^2 + f^\sharp & \text{in } \Omega^\sharp \\ v = 0 & \text{on } \partial\Omega^\sharp \end{cases}$$

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Theorem If there exists a bounded solution to the symmetrized problem, then there exists a solution to the given problem.

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Theorem If there exists a bounded solution to the symmetrized problem, then there exists a solution to the given problem.

[L. Boccardo - F. Murat - J.P. Puel, 1982, . . . ,1992], [J.M. Rakotoson, 1987], [A. Alvino - P.-L. Lions - G. Trombetti, 1990], [V.F. - F. Murat, 2000], [V.F. - B. Messano, 2007], . . .

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The existence of the solution to the symmetrized problem can be obtained under "smallness" assumptions on f.
Indeed, if v is a bounded solution of problem:

$$\begin{cases} -\Delta v = |Dv|^2 + f^\sharp & \text{in } \Omega^\sharp \\ v = 0 & \text{on } \partial\Omega^\sharp \end{cases}$$

putting $w = e^v - 1$, we have:

$$\begin{cases} -\Delta w = wf^\sharp + f^\sharp & \text{in } \Omega^\sharp \\ w = 0 & \text{on } \partial\Omega^\sharp. \end{cases}$$

A generalized notion of perimeter

Let $H : \mathbb{R}^n \to [0, +\infty[$ be a convex function satisfying the following properties:

$$H(tx) = |t|H(x), \qquad \forall x \in \mathbb{R}^n, \ \forall t \in \mathbb{R}.$$

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with $0 < \alpha \leq \beta$.

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with $0 < \alpha \leq \beta$.

Furthermore, let us suppose that the convex set

$$K = \{x \in \mathbb{R}^n : H(x) \leq 1\}$$

has measure $|K|$ equal to ω_n.
Sometimes H is called the "gauge" of K.

A generalized notion of perimeter

The support function of K

$$H^o(x) = \sup_{\xi \in K} < x, \xi >,$$

is a convex function such that H and H^o are one the polar function of the other, that is,

$$H^o(x) = \sup_{\xi \neq 0} \frac{< x, \xi >}{H(\xi)} \quad \text{and} \quad H(x) = \sup_{\xi \neq 0} \frac{< x, \xi >}{H^o(\xi)}.$$

Clearly $H^o(x)$ is the gauge of

$$K^o = \{x \in \mathbb{R}^n : H^o(x) \leq 1\}.$$

We denote by k_n the measure of K^o.

A generalized notion of perimeter

Let $E \subset \mathbb{R}^n$ be a measuble set of finite measure. It is possible to give a "generalized" definition of perimeter of E with respect to H such that, for sufficiently regular domains, it results:

$$Per_H(E) = \int_{\partial E} H(\nu)\, d\mathcal{H}^{n-1}(x)$$

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The following *isoperimetric inequality* (Wulff Theorem) holds true

$$Per_H(E) \geq n k_n^{1/n} |E|^{1-1/n}.$$

Equality hold if and only if E is a sublevel of H^o (Wulff shape), modulo translations..

[J.E. Taylor, 1979, 1988, . . . , 2002], [I. Fonseca - S. Müller, 1991], [B. Dacorogna - C.-E. Pfister, 1992], [M. Amar -G. Bellettini, 1994], [A. Alvino - V. F. - P.-L. Lions - G. Trombetti, 1997]

A problem with symmetric data

$$\begin{cases} -\displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big((H(Du))^{n-1} H_{\xi_i}(Du)\Big) = f(u) & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega = \{x \in \mathbb{R}^n : H^o(x) < R\}$, with $R > 0$, $n \geq 2$, has the Wulff shape.

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We suppose that $f : \mathbb{R} \to \mathbb{R}$ is such that $f(s) > 0$ and $f(s) \leq c_1|s|^r + c_2$ for $s > 0$ and for some $r > 0$.

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Theorem If H is strictly convex, $H(\xi)^n$ is of class $C^2(\mathbb{R}^n)$ and f satisfies the above assumptions, then the level sets of any positive solution to the given problem have the Wulff shape associated to H and then are homothetic to Ω.

[B. Gidas - W. Ni - L. Nirenberg, 1979], [P.-L. Lions, 1981], [S. Kesavan - F. Pacella, 1994], [M. Belloni - V.F.- B. Kawohl, 2003]

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[J. Serrin, 1971]

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Above result has been proven with a new approach based also on symmetrization methods and it has been extended to the case of hessian operators.
[B. Brandolini - C. Nitsch - P. Salani - C. Trombetti, 2007]