

Elliptic equations with first order terms

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1 Introduction

The aim of this short course is to take a look to some questions arising around elliptic equations with first order terms. The following models are the main examples to be kept in mind. The domain Ω will always be assumed to be a bounded subset of \mathbf{R}^N .

1. The linear equation

$$-\Delta u + \lambda u = b(x) \cdot \nabla u + f(x) \quad x \in \Omega \quad (1.1)$$

where $\lambda \geq 0$. This model arises e.g. as the stationary equation of diffusion-advection problems

$$u_t - \Delta u + \lambda u = b(x) \cdot \nabla u + f(x).$$

The example (1.1) can be considered in more generality with inhomogeneous diffusion, e.g. for the equation

$$-\operatorname{div}(A(x)\nabla u) + \lambda u = b(x) \cdot \nabla u + f(x) \quad (1.2)$$

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where $A(x)$ is a coercive and bounded matrix, i.e. $\alpha I \leq A(x) \leq \beta I$ for a.e. $x \in \Omega$. The study of (1.2) is reflected in its dual equation

$$-\operatorname{div}(A(x)^*\nabla u) + \lambda u = \operatorname{div}(b(x)u) + f(x) \quad (1.3)$$

where A^* is the adjoint equation. This last model is the stationary equation of convection-diffusion models appearing frequently in connection with conservation laws. Indeed, the structure of (1.3) strongly relies on the divergence form.

Although (1.2) and (1.3) are dual equations (which implies that, using linear theory, one can use the results on one model to deduce results for the other), they have different features; when we consider nonlinear generalizations of such models, the approach for the two types could be quite different. In this course, we will restrict to generalizations of (1.2).

On the other hand, the equation (1.1) also arises as the Kolmogorov equation related to stochastic diffusion processes

$$dX_t = b(X_t)dt + \sqrt{2}dW_t, \quad X_0 = x \in \Omega,$$

where W_t is a standard Wiener process (so-called Brownian motion). In particular, if (1.1) is complemented with the Dirichlet condition on $\partial\Omega$, then we have

$$u(x) = E_x \left\{ \int_0^{\tau_x} [f(X_t)] e^{-\lambda t} dt \right\}$$

where E_x is the conditional expectation with respect to $X_0 = x$ and τ_x is the first exit time from Ω .

Observe that in this situation considering inhomogeneous diffusions (i.e. $\sigma(X_t)dW_t$ instead of the pure Wiener process) would lead to the Kolmogorov equation

$$-Tr(A(x)D^2u) + \lambda u = b(x) \cdot \nabla u + f(x)$$

where $A(x) = \sigma(x)\sigma^T(x)$ and $Tr(M)$ denotes the trace of the matrix M . This equation is not in divergence form and, unless A is Lipschitz continuous, can not be reduced to a structure like (1.2). A proper study of such models in non-divergence form would need the introduction of viscosity solutions, which we will mention only later on.

2. The nonlinear model

$$-\operatorname{div}(A(x)\nabla u) + \lambda u = |\nabla u|^q + f(x) \quad (1.4)$$

where $q > 1$. Such kind of models may appear in several applications. Here the superlinear growth with respect to $|\nabla u|$ provides new interesting questions, which we will address in Section 3.

Similar models may also arise in connection with stochastic equations subject to control processes. The simplest model is when we consider the SDE

$$dX_t = b(X_t, \alpha)dt + \sqrt{2}dW_t, \quad X_0 = x \in \Omega,$$

controlled by the process α , with a given cost criterion

$$E_x \left\{ \int_0^{\tau_x} [L(X_t, \alpha)] e^{-\lambda t} dt \right\}$$

The dynamic programming principle due to Bellmann implies that the so-called value function

$$u(x) = \min_{a \in \mathcal{A}} E_x \left\{ \int_0^{\tau_x} [L(X_t, \alpha)] e^{-\lambda t} dt \right\}$$

where \mathcal{A} is the class of admissible controls, solves the equation

$$-\Delta u + \lambda u + H(x, \nabla u) = 0$$

with Dirichlet boundary conditions, where $H(x, p) = \sup_{a \in \mathcal{A}} [b(x, \alpha) \cdot p - L(x, \alpha)]$. Note that in this model the function $H(x, p)$ (called the Hamiltonian) is convex in p .

A very important case is given by the so-called viscous Hamilton-Jacobi equation

$$-\varepsilon \Delta u + \lambda u + |\nabla u|^m = f(x), \quad (1.5)$$

where $\varepsilon > 0$. This model is often introduced as an approximation of the eikonal-type equation

$$\lambda u + |\nabla u|^m = f(x)$$

The limit $\varepsilon \rightarrow 0$ (vanishing viscosity limit) gives the name and one of the pioneering applications of the theory of viscosity solutions of first order problems.

A short plan of the course will be the following.

In Section 2 we review the case when the first order term has linear (or sublinear) growth with respect to $|\nabla u|$, showing some nonlinear methods which can be used to obtain the existence of solutions, the weak maximum principle and then uniqueness.

In Section 3 we discuss the new problems given when the growth is superlinear: possible failure of uniqueness in $H_0^1(\Omega)$, and possible failure of existence when the equation has no zero order terms (i.e. $\lambda = 0$ in (1.4)). In the simple case when $f \in L^\infty(\Omega)$ and $\lambda > 0$, we give an existence and uniqueness result for bounded solutions, showing some ideas adapted to the superlinear growth.

In Section 4 we consider the model case of viscous Hamilton-Jacobi equation with $f \in L^\infty(\Omega)$ and we discuss what happens in the singular limit $\lambda \rightarrow 0$ (ergodic limit). A basic tool for that is the pointwise gradient bound in Lemma 4.1. This is also a crucial tool when performing the other singular limit $\varepsilon \rightarrow 0$ (vanishing viscosity limit), which I will mention at the end if time will be left.

2 The case of linear growth in the $H_0^1(\Omega)$ setting

Consider the linear problem

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) + \lambda u = b(x) \cdot \nabla u + f(x) & \text{in } \Omega, \\ u \in H_0^1(\Omega) \end{cases} \quad (2.1)$$

where we assume that $A(x)$ is a measurable matrix such that

$$A(x) \in L^\infty(\Omega)^{N \times N}, \quad A(x)\xi\xi \geq \alpha|\xi|^2, \quad (2.2)$$

for some $\alpha > 0$, and

$$f \in H^{-1}(\Omega) \quad (2.3)$$

$$b \in L^N(\Omega)^N. \quad (2.4)$$

Observe that the assumption $b \in L^N(\Omega)^N$ implies that, for every $u \in H_0^1(\Omega)$, we have that $b \cdot \nabla u \in H^{-1}(\Omega)$. Indeed, through Hölder inequality, one can prove that $b \cdot \nabla u \in L^{\frac{2N}{N+2}}(\Omega)$, and then recall that $L^{\frac{2N}{N+2}}(\Omega) \subset H^{-1}(\Omega)$ because of Sobolev inequality.

Even if one considers the linear case, it is easy to realize that the operator $A(u) = -\operatorname{div}(A(x)\nabla u) + \lambda u - b(x) \cdot \nabla u$ is not coercive in general, unless we have $\|b\|_{L^N(\Omega)}$ small enough. This is why the existence of solutions of (2.1) cannot be simply deduced from the Lax-Milgram theorem (concerning coercive bounded bilinear forms).

By contrast, we are going to see how some nonlinear methods allow one to prove the existence and uniqueness of solutions. In order not to be too stick to the linear example, let us consider a slightly more general case

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) + \lambda u = H(x, \nabla u) + f(x) & \text{in } \Omega, \\ u \in H_0^1(\Omega) \end{cases} \quad (2.5)$$

where $H(x, \xi)$ satisfies

$$|H(x, \xi)| \leq |b(x)| (|\xi| + 1) \quad (2.6)$$

and

$$|H(x, \xi) - H(x, \eta)| \leq |b(x)| |\xi - \eta|. \quad (2.7)$$

We have then

Theorem 2.1 *Assume (2.2), (2.3), and (2.6), (2.7), where b satisfies (2.4). Let $\lambda \geq 0$. Then there exists a unique solution of (2.5).*

Let us first prove the uniqueness part of the above result. This is a consequence of the validity of the following *weak maximum principle in $H_0^1(\Omega)$* (see also Chapter 8 in [16]).

Proposition 2.1 *Let $w \in H_0^1(\Omega)$ satisfy*

$$-\operatorname{div}(A(x)\nabla w) + \lambda w \leq |b(x)| |\nabla w| \quad \text{in } \Omega,$$

where $b \in L^N(\Omega)$. Then $w \leq 0$.

Proof. For $k > 0$, take $(w - k)^+$ as test function and obtain

$$\int_{\Omega} A(x)\nabla w \nabla (w - k)^+ dx \leq \int_{\Omega} |b(x)| |\nabla w| (w - k)^+ dx$$

which implies, setting $w_k = (w - k)^+$,

$$\int_{\Omega} A(x)\nabla w_k \nabla w_k dx \leq \int_{\Omega} |b(x)| |\nabla w_k| w_k dx$$

Using (2.2) and Hölder inequality with exponents $(N, 2, \frac{2N}{N-2})$, one gets

$$\alpha \int_{\Omega} |\nabla w_k|^2 dx \leq \left(\int_{E_k} |b|^N dx \right)^{\frac{1}{N}} \left(\int_{\Omega} |\nabla w_k|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} w_k^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{2N}},$$

where

$$E_k = \{x \in \Omega : w(x) > k, \quad |\nabla w(x)| > 0\}.$$

Sobolev inequality implies

$$\alpha \int_{\Omega} |\nabla w_k|^2 dx \leq S \left(\int_{E_k} |b|^N dx \right)^{\frac{1}{N}} \left(\int_{\Omega} |\nabla w_k|^2 dx \right), \quad (2.8)$$

where S is the Sobolev constant.

Now argue by contradiction, assuming that $\sup w > 0$. Set $M = \sup w$; then, either $M = +\infty$, and then clearly

$$\lim_{k \rightarrow M} \text{meas}(E_k) = 0, \quad (2.9)$$

or M is finite, and in that case, by properties of functions in Sobolev spaces, we have that $|\nabla w(x)| = 0$ a.e. on $\{w(x) = M\}$, hence we still deduce (2.9). Therefore, since (2.9) holds true, we have $\|b\|_{L^N(E_k)} \rightarrow 0$, hence there exists $k_0 < M$ such that $\|B\|_{L^N(E_k)} < \frac{\alpha}{2S}$ for $k \geq k_0$. We deduce from (2.8) that $w \leq k_0$ a.e., hence $\sup w \leq k_0 < M$, getting a contradiction. ■

The above result extends the classical maximum principle to the $H_0^1(\Omega)$ setting. As a consequence we have the comparison principle in $H_0^1(\Omega)$.

Corollary 2.1 *Assume (2.2), (2.3), and (2.6), (2.7), where b satisfies (2.4). Let $\lambda \geq 0$. Let u_1, u_2 be respectively a subsolution and a supersolution of*

$$-\text{div}(A(x)\nabla u) + \lambda u = H(x, \nabla u) + f(x) \quad \text{in } \Omega$$

such that $u_1 \leq u_2$ on $\partial\Omega$ (i.e. $(u_1 - u_2)^+ \in H_0^1(\Omega)$). Then we have $u_1 \leq u_2$ in Ω .

Proof. Set $w = (u_1 - u_2)$. Then w satisfies, thanks to (2.7):

$$-\text{div}(A(x)\nabla w) + \lambda w = H(x, \nabla u_1) - H(x, \nabla u_2) \leq |b(x)| |\nabla w| \quad \text{in } \Omega.$$

Now, $w^+ \in H_0^1(\Omega)$ and satisfies the same inequality. Applying Proposition 2.1 we deduce that $w \leq 0$, hence $u_1 \leq u_2$. ■

We deduce obviously from the above result the uniqueness part of Theorem 2.1. The existence part can be obtained with several possible approaches. We will use a fixed point argument.

Proposition 2.2 Assume (2.2), (2.3), (2.6), (2.7) where b satisfies (2.4). Let $\lambda \geq 0$. For any $v \in H_0^1(\Omega)$, define the operator $T : H_0^1(\Omega) \mapsto H_0^1(\Omega)$ such that

$$T(v) = u \iff \begin{cases} -\operatorname{div}(A(x)\nabla u) + \lambda u = H(x, \nabla v) + f(x) & \text{in } \Omega, \\ u \in H_0^1(\Omega) \end{cases}$$

Then

- (i) T is continuous
- (ii) T is compact (i.e. transforms bounded sets into precompact sets)
- (iii) There exists a constant $M > 0$ such that

$$\|u\|_{H_0^1(\Omega)} \leq M$$

for every $u \in H_0^1(\Omega)$ and every $\sigma \in [0, 1]$ such that $u = \sigma T(u)$.

Proof.

The continuity of T is a simple exercise. Let us prove that T is compact. Take a sequence v_n which is bounded in $H_0^1(\Omega)$ and let $u_n = T(v_n)$. Since $|b||\nabla v_n|$ is bounded in $H^{-1}(\Omega)$, we easily get that u_n is bounded in $H_0^1(\Omega)$, hence, by Rellich theorem, it is relatively compact in $L^2(\Omega)$ and for the almost everywhere convergence in Ω . Without relabeling the index, we still call u_n the (sub)sequence which strongly converges in $L^2(\Omega)$ and a.e. to some function $u \in H_0^1(\Omega)$, and such that ∇u_n converges to ∇u weakly in $L^2(\Omega)$.

Now take $u_n - u$ as test function to get

$$\int_{\Omega} A(x)\nabla u_n \cdot \nabla (u_n - u) dx + \lambda \int_{\Omega} u_n (u_n - u) dx = \int_{\Omega} H(x, \nabla v_n)(u_n - u) + \langle f, u_n - u \rangle,$$

and then

$$\begin{aligned} \int_{\Omega} A(x)\nabla (u_n - u) \cdot \nabla (u_n - u) dx &= \int_{\Omega} H(x, \nabla v_n)(u_n - u) \\ &+ \langle f, u_n - u \rangle - \int_{\Omega} A(x)\nabla u \nabla (u_n - u) dx - \lambda \int_{\Omega} u_n (u_n - u) dx. \end{aligned}$$

Last three terms go to zero by weak convergence of ∇u_n and strong convergence of u_n , hence, using also the coercivity,

$$\alpha \int_{\Omega} |\nabla (u_n - u)|^2 dx \leq \int_{\Omega} H(x, \nabla v_n)(u_n - u) dx + \varepsilon_n$$

where $\varepsilon_n \rightarrow 0$. We also have, for any $k > 0$,

$$\begin{aligned} \int_{\Omega} H(x, \nabla v_n)(u_n - u) dx &= \int_{\Omega} H(x, \nabla v_n) T_k(u_n - u) dx \\ &+ \int_{\{|u_n - u| > k\}} |b(x)| |\nabla v_n| |u_n - u| dx \end{aligned}$$

where $T_k(s) = \max(-k, \min(s, k))$ denotes the truncation at levels $\pm k$. By Hölder inequality and since u_n is bounded in $L^{2^*}(\Omega)$, v_n is bounded in $H_0^1(\Omega)$, we get

$$\alpha \int_{\Omega} |\nabla(u_n - u)|^2 dx \leq \int_{\Omega} H(x, \nabla v_n) T_k(u_n - u) dx + C_0 \left(\int_{\{|u_n - u| > k\}} |b(x)|^N dx \right)^{\frac{1}{N}}$$

The second integral goes to zero because $T_k(u_n - u) \rightarrow 0$ strongly in any $L^p(\Omega)$. Thus we take the limit for $n \rightarrow \infty$ obtaining

$$\lim_{n \rightarrow \infty} \alpha \int_{\Omega} |\nabla(u_n - u)|^2 dx \leq C_0 \sup_n \left(\int_{\{|u_n - u| > k\}} |b(x)|^N dx \right)^{\frac{1}{N}}$$

Observe now that the set $\{|u_n - u| > k\}$ has small measure uniformly with respect to n (because $u_n - u$ is bounded in $L^2(\Omega)$), hence taking the limit as $k \rightarrow \infty$ we find that

$$\alpha \int_{\Omega} |\nabla(u_n - u)|^2 dx \rightarrow 0$$

hence u_n is strongly convergent, i.e. compact, in $H_0^1(\Omega)$.

Let us prove now (iii). Argue by contradiction, hence assume that for any n you have $u_n \in H_0^1(\Omega)$, $\sigma_n \in [0, 1]$ such that $u_n = \sigma_n T(u_n)$ and $\|u_n\|_{H_0^1(\Omega)} \geq n$. This means that

$$\|u_n\|_{H_0^1(\Omega)} \rightarrow \infty.$$

Having $u_n = \sigma_n T(u_n)$ means that

$$-\operatorname{div}(A(x) \nabla u_n) + \lambda u_n = \sigma_n H(x, \nabla u_n) + \sigma_n f(x)$$

Set now $w_n = \frac{u_n}{\|u_n\|_{H_0^1(\Omega)}}$, hence we get

$$-\operatorname{div}(A(x) \nabla w_n) + \lambda w_n = \sigma_n \frac{H(x, \nabla u_n)}{\|u_n\|_{H_0^1(\Omega)}} + \sigma_n \frac{f(x)}{\|u_n\|_{H_0^1(\Omega)}}$$

Since $\sigma_n \leq 1$, and since

$$\left| \frac{H(x, \nabla u_n)}{\|u_n\|_{H_0^1(\Omega)}} \right| \leq |b(x)| \left(|\nabla w_n| + \frac{1}{\|u_n\|_{H_0^1(\Omega)}} \right)$$

using that w_n is bounded in $H_0^1(\Omega)$ (actually $\|w_n\|_{H_0^1(\Omega)} \equiv 1$), we have that the right hand side is bounded in $H^{-1}(\Omega)$. Therefore, as in the previous step we can deduce that w_n is relatively compact in $H_0^1(\Omega)$, so there exists a subsequence (still called w_n) and $w \in H_0^1(\Omega)$ such that $w_n \rightarrow w$ strongly in $H_0^1(\Omega)$.

We deduce from the equation

$$\begin{aligned} -\operatorname{div}(A(x)\nabla w_n) + \lambda w_n &= \sigma_n \frac{H(x, \nabla u_n)}{\|u_n\|_{H_0^1(\Omega)}} + \sigma_n \frac{f(x)}{\|u_n\|_{H_0^1(\Omega)}} \\ &\leq |b(x)| \left(|\nabla w_n| + \frac{1}{\|u_n\|_{H_0^1(\Omega)}} \right) + \sigma_n \frac{f(x)}{\|u_n\|_{H_0^1(\Omega)}} \end{aligned}$$

and letting $n \rightarrow \infty$ we get

$$-\operatorname{div}(A(x)\nabla w) + \lambda w \leq |b(x)| |\nabla w|.$$

Proposition 2.1 implies that $w \leq 0$. Reasoning in the same way for $-w$ we obtain that $w \geq 0$, hence $w = 0$. However $\|w_n\|_{H_0^1(\Omega)} = 1$ and since w_n is strongly convergent we get a contradiction. ■

We are ready now to prove the basic existence and uniqueness result. We only need to recall the following classical fixed point theorem (see Theorem 11.3 in [16]).

Theorem 2.2 *Let X be a Banach space and $T : X \rightarrow X$ a map which is continuous, compact and such that there exists a constant $M > 0$: $\|u\|_X \leq M$ for every $u \in X$ and every $\sigma \in [0, 1]$ satisfying $u = \sigma T(u)$. Then T has a fixed point in X , i.e. there exists $u \in X$ such that $T(u) = u$.*

Proof of Theorem 2.1. The existence of a solution follows from Theorem 2.2 using Proposition 2.2. Uniqueness immediately follows from Corollary 2.1. ■

Observe that the proof of property (iii) in Proposition 2.2 implies an *a priori* estimate for problem (2.5). Indeed, if we apply the above estimate to

$\frac{u}{\|f\|_{H^{-1}(\Omega)}}$, we have proved that under assumptions (2.2), (2.6), (2.3), there exists a constant $C_b > 0$ such that any $u \in H_0^1(\Omega)$ solution of (2.5) satisfies

$$\|u\|_{H_0^1(\Omega)} \leq C_b \|f\|_{H^{-1}(\Omega)}$$

The constant C_b depends on b but not on f and u . Actually, it is worth saying that a stronger, more significant, version of this a priori estimate holds and is the following (the proof is not trivial and uses refined truncation methods, see [10], or may be obtained using rearrangements, see e.g. [13]).

Theorem 2.3 *Assume (2.2), (2.6), (2.4), (2.3), let $\lambda \geq 0$. Then there exist constants C_0, C_1 (independent on b, f) such that any u solution of (2.5) satisfies*

$$\|u\|_{H_0^1(\Omega)} \leq C_0 e^{C_1 \|b\|_{L^N(\Omega)}} \|f\|_{H^{-1}(\Omega)}.$$

3 The case with superlinear growth.

In the previous section we have seen that when the first order terms grow at most linearly, then we have existence and uniqueness in $H_0^1(\Omega)$, the validity of the weak maximum principle and a priori estimates. Moreover, all the above results are true both for $\lambda > 0$ and for $\lambda = 0$. Now we consider the case of superlinear growth, in particular the simple model

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) + \lambda u = \gamma|\nabla u|^q + f(x) & \text{in } \Omega, \\ u \in H_0^1(\Omega) \end{cases} \quad (3.1)$$

where $1 < q \leq 2, \gamma > 0$.

3.1 Some examples showing new difficulties

Let us soon observe some new phenomena which make this situation different.

Example 3.1 *(uniqueness may fail in $H_0^1(\Omega)$)* Let Ω be the unit ball in \mathbb{R}^N , and let $\frac{N}{N-1} < q < 2$. The function $u(x) = M(|x|^{-\alpha} - 1)$, with $\alpha = \frac{2-q}{q-1}$ and $M = \frac{[N-1-(\alpha+1)]^{\frac{1}{q-1}}}{\alpha}$, satisfies (in the sense of distributions)

$$\begin{cases} -\Delta u = |\nabla u|^q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, we have $u \in H_0^1(\Omega)$ if $q > 1 + \frac{2}{N}$. A similar example holds when $q = 2$ using $u = M|\log|x||$.

The previous example shows that when $q > 1 + \frac{2}{N}$ the comparison principle (and the weak maximum principle too) in $H_0^1(\Omega)$ does not hold, and we cannot have uniqueness in $H_0^1(\Omega)$. Note that here we have $f = 0$, so this is not due to the regularity of f . The problem is the class of solutions, and $u \in H_0^1(\Omega)$ may be no more convenient.

Let us remark why the value $q = 1 + \frac{2}{N}$ is important. If you write

$$-\Delta u = |\nabla u|^{q-2} \nabla u \nabla u \quad (3.2)$$

you have

$$-\Delta u = B(x) \nabla u \quad \text{where } B(x) = |\nabla u|^{q-2} \nabla u. \quad (3.3)$$

In order to have uniqueness, you would need $|B| \in L^N(\Omega)$ as in Proposition 2.1. But if the solution $u \in H_0^1(\Omega)$, we only have $|B| \in L^{\frac{2}{q-1}}(\Omega)$, hence $|B| \in L^N(\Omega)$ only if $\frac{2}{q-1} \geq N$ which means $q \leq 1 + \frac{2}{N}$.

In fact, the case $q \leq 1 + \frac{2}{N}$ turns out to be simpler, and it is possible to prove that, for any $f \in H^{-1}(\Omega)$, problem (3.1) has at most one solution in $H_0^1(\Omega)$ (see [6], [7]).

However, the uniqueness is not the only problem: because of superlinear growth, existence and a priori estimates also give new difficulties, as shown in the following examples.

Example 3.2 (*existence may fail when $\lambda = 0$*) Consider the problem

$$\begin{cases} -\Delta u = |\nabla u|^2 + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.4)$$

Set $v = e^u - 1$, then v solves the problem

$$\begin{cases} -\Delta v = f(x)(1 + v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Let λ_1 be the first eigenvalue of the Laplacian (with Dirichlet conditions) in Ω . If you have $f(x) > \lambda_1$, this implies that $f \geq 0$, hence $u \geq 0$. Then v is a positive function such that

$$-\Delta v = f(x)(1 + v) \geq \lambda_1 v$$

which is impossible due to the properties of the first eigenvalue. This shows that problem (3.4) cannot have solutions. (A more detailed analysis of the existence and nonexistence of solutions for this particular problem can be found in [1], [15])

Example 3.3 (necessary conditions on f are needed when $\lambda = 0$)

Instead of (3.4) consider the more general problem

$$\begin{cases} -\Delta u = |\nabla u|^q + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.5)$$

where $1 < q \leq 2$ and $f \geq 0$.

Assume that a solution exists, and take $\varphi^{q'}$ as test function, where $\varphi \in C_c^\infty(\Omega)$ ($q' = \frac{q}{q-1}$). Then we get

$$q' \int_{\Omega} \nabla u \nabla \varphi \varphi^{q'-1} dx = \int_{\Omega} |\nabla u|^q \varphi^{q'} dx + \int_{\Omega} f(x) \varphi^{q'} dx.$$

Since Young's inequality implies, for some constant C_q ,

$$q' \nabla u \nabla \varphi \varphi^{q'-1} \leq |\nabla u|^q \varphi^{q'} + C_q |\nabla \varphi|^{q'}$$

we deduce that f must satisfy

$$\int_{\Omega} f(x) \varphi^{q'} dx \leq C_q \int_{\Omega} |\nabla \varphi|^{q'} dx \quad \forall \varphi \in C_c^\infty(\Omega). \quad (3.6)$$

This condition is not satisfied by any f and gives precise restrictions on the data f for which (3.5) admits a solution.

These necessary conditions on f are investigated in a deep way in [19] (see also [2]). Here let us observe that (3.6) gives two kind of restrictions, both on the size of f and on the regularity of f . For example, let $f \equiv f_0$ be a constant. If you fix some cut-off function φ in Ω , you observe from (3.6) that f_0 cannot be arbitrarily large, but it should be bounded by some quantity depending only on q and Ω (this is like in Example 3.2, where the condition found was that $f \leq \lambda_1$): this is a size condition on f . On the other hand, if you localize condition (3.6) on small balls B_R (using suitable cut-off functions φ with $\varphi = 1$ on B_R , $\varphi = 0$ outside B_{2R}), you observe that f must satisfy, for some constant K

$$\int_{B_R} f dx \leq K R^{N-q'}, \quad \text{for every ball } B_R \subset\subset \Omega.$$

This is a regularity condition, namely this implies that f belongs to the so-called Morrey space of order $\frac{N}{q'}$, denoted by $M^{\frac{N}{q'}}(\Omega)$ (see [16] for Morrey spaces). It is a good exercise to verify that, when $q > 1 + \frac{2}{N}$, there are

functions belonging to $H^{-1}(\Omega)$ but not satisfying this condition. Indeed, in terms of Lebesgue spaces, one has

$$L^m(\Omega) \subset M^{\frac{N}{q'}}(\Omega)$$

only if $m \geq \frac{N}{q'}$.

For a discussion of the consequence of these necessary conditions, according to the different values of q , when f belongs to some Lebesgue space, you may see the introduction in [6], [18] or [2]. A deep and complete analysis of (3.6) (and the equivalence with capacity criteria) can be found in [19] and in related works by V. Maz'ja, I.E. Verbitsky.

On account of previous examples, we derive the following comments. When $q \leq 1 + \frac{2}{N}$, it is still possible to work with $f \in H^{-1}(\Omega)$ and solutions $u \in H_0^1(\Omega)$, since weak comparison holds in $H_0^1(\Omega)$. On the other hand when $q > 1 + \frac{2}{N}$ it is necessary to work in different spaces, both for the data f and for the solutions u (since the comparison principle fails in $H_0^1(\Omega)$). Moreover, when $\lambda = 0$, in both cases the existence requires some size condition on the data.

A detailed study of problem (3.1) in both ranges $q \leq 1 + \frac{2}{N}$ and $1 + \frac{2}{N} < q < 2$ can be found in [17], [18], [14], [6], with both existence and uniqueness results when f belongs to the suitable Lebesgue space (the natural choice is to take $f \in L^m(\Omega)$ with $m \geq \frac{N}{q'}$). The case $q = 2$ is a bit special and the reader is referred to [9], [12], [15], for the existence part and to [4], [5], [1] for some uniqueness (or nonuniqueness) results. On the other hand, the growth $q > 2$ makes the problem completely different and usually requires a different approach (see e.g. [21], [11]).

Let us now concentrate on the simple case that $f \in L^\infty(\Omega)$ and $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

3.2 A simple case: $f \in L^\infty(\Omega)$ and bounded solutions.

We consider here problem (3.1) where $1 \leq q \leq 2$, and when $f \in L^\infty(\Omega)$ and $\lambda > 0$. This assumption on λ allows us to avoid the restrictions on the size of f (see Examples 3.2 and 3.3). In particular we have a standard estimate as in the weak maximum principle.

Theorem 3.1 *Let $f \in L^\infty(\Omega)$, and let $\lambda > 0$. If $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ satisfies*

$$-\operatorname{div}(A(x)\nabla u) + \lambda u \leq \gamma|\nabla u|^q + f(x) \quad \text{in } \Omega$$

then

$$\sup_{\Omega} u \leq \frac{\|f\|_{\infty}}{\lambda}.$$

In particular, any solution of (3.1) satisfies

$$\|u\|_{\infty} \leq \frac{\|f\|_{\infty}}{\lambda} \quad (3.7)$$

Proof.

Set $k_0 = \frac{\|f\|_{\infty}}{\lambda}$ and $w = u - k_0$. Then, subtracting λk_0 in equation (3.1) we obtain that w satisfies

$$-\operatorname{div}(A(x)\nabla w) + \lambda w \leq \gamma|\nabla w|^q,$$

and of course w^+ still belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$. The conclusion of the Theorem follows once we prove that $w \leq 0$.

Consider first the case $1 + \frac{2}{N} < q < 2$. We use a similar strategy as in Proposition 2.1. Use now $[(w - k)^+]^{2\sigma-1}$ as test function, where σ will be chosen later. We obtain

$$(2\sigma - 1) \int_{\Omega} A(x)\nabla w \nabla (w - k)^+ [(w - k)^+]^{2\sigma-2} dx \leq \gamma \int_{\Omega} |\nabla w|^q [(w - k)^+]^{2\sigma-1} dx$$

which implies, setting $w_k = (w - k)^+$,

$$(2\sigma - 1) \int_{\Omega} A(x)\nabla w_k \nabla w_k w_k^{2\sigma-2} dx \leq \gamma \int_{\Omega} |\nabla w|^q w_k^{(\sigma-1)q} w_k^{2\sigma-1-(\sigma-1)q} dx.$$

Using (2.2) and Hölder inequality we get

$$\begin{aligned} & \alpha(2\sigma - 1) \int_{\Omega} |\nabla w_k|^2 w_k^{2\sigma-2} dx \leq \\ & \leq \gamma \left(\int_{\Omega} |\nabla w_k|^2 w_k^{2\sigma-2} dx \right)^{\frac{q}{2}} \left(\int_{\Omega} w_k^{[2\sigma-1-(\sigma-1)q]\frac{2}{2-q}} \right)^{1-\frac{q}{2}}. \end{aligned} \quad (3.8)$$

Now choose σ such that

$$[2\sigma - 1 - (\sigma - 1)q]\frac{2}{2 - q} = \sigma 2^*.$$

This gives $\sigma = \frac{(N-2)(q-1)}{2(2-q)}$ (you may check that $\sigma > 1$ as soon as $q > 1 + \frac{2}{N}$). We obtain then from Sobolev inequality

$$\left(\int_{\Omega} w_k^{[2\sigma-1-(\sigma-1)q]\frac{2}{2-q}} \right) = \left(\int_{\Omega} w_k^{\sigma 2^*} dx \right) \leq S \left(\int_{\Omega} |\nabla(w_k^{\sigma})|^2 dx \right)^{\frac{N}{N-2}}.$$

Therefore we conclude from (3.8)

$$\alpha(2\sigma - 1) \int_{\Omega} |\nabla(w_k^{\sigma})|^2 dx \leq \gamma S \left(\int_{\Omega} |\nabla(w_k^{\sigma})|^2 dx \right)^{\frac{q}{2} + \frac{N}{N-2}(1-\frac{q}{2})},$$

which implies

$$\alpha(2\sigma - 1) \leq \gamma S \left(\int_{\Omega} |\nabla(w_k^{\sigma})|^2 dx \right)^{\frac{2-q}{N-2}}.$$

We conclude as in Proposition 2.1. Indeed, should $M = \sup_{\Omega} w$ be positive, then we take $0 < k < M$. Since last integral only concerns the set $E_k = \{x \in \Omega : w(x) > k, |\nabla w(x)| > 0\}$, and since $|\nabla w| = 0$ almost everywhere on the set $\{w(x) = M\}$, when $k \uparrow M$ the right hand side will converge to zero, obtaining a contradiction.

The case $q = 2$ is similar, but we multiply by $e^{\frac{\gamma}{\alpha}w^+} - 1$ and we obtain

$$\frac{\gamma}{\alpha} \int_{\Omega} A(x) \nabla w \nabla w^+ e^{\frac{\gamma}{\alpha}w^+} dx \leq \gamma \int_{\Omega} |\nabla w|^2 (e^{\frac{\gamma}{\alpha}w^+} - 1) dx$$

hence using (2.2) we deduce that $w \leq 0$.

Finally, if $q \leq 1 + \frac{2}{N}$ we write the right hand side as in (3.2), (3.3), hence we can apply Proposition 2.1 to conclude. \blacksquare

Remark 3.1 *The result of Theorem 3.1 is not trivial since it would not hold assuming only $u \in H_0^1(\Omega)$, when $q > 1 + \frac{2}{N}$. In particular, modifying a bit the Example 3.1 one can construct some u which is not negative and satisfies*

$$-\Delta u + \lambda u \leq |\nabla u|^q \quad \text{in } \Omega,$$

with the property, if $q > 1 + \frac{2}{N}$, that $u \in H_0^1(\Omega)$. This is possible for any $\lambda > 0$.

As a corollary of the previous estimate, we derive the following comparison principle.

Corollary 3.1 *Assume that $f \in L^\infty(\Omega)$, and let $\lambda > 0$. Let $u, v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be respectively a subsolution and a supersolution of (3.1). Then we have that $u \leq v$.*

Proof.

Set $u_\varepsilon = (1 + \varepsilon)u$. Then u_ε satisfies

$$-\operatorname{div}(A(x)\nabla u_\varepsilon) + \lambda u_\varepsilon = \gamma(1 + \varepsilon)|\nabla u|^q + (1 + \varepsilon)f(x)$$

and subtracting the equation of v we obtain

$$-\operatorname{div}(A(x)\nabla w_\varepsilon) + \lambda w_\varepsilon = \gamma(1 + \varepsilon)|\nabla u|^q - \gamma|\nabla v|^q + \varepsilon f(x)$$

where $w_\varepsilon = u_\varepsilon - v$. Now observe that since

$$\nabla u = \frac{1}{1 + \varepsilon}\nabla v + \frac{\varepsilon}{1 + \varepsilon}\frac{\nabla(u_\varepsilon - v)}{\varepsilon}$$

by convexity we have

$$|\nabla u|^q \leq \frac{1}{1 + \varepsilon}|\nabla v|^q + \frac{\varepsilon}{1 + \varepsilon}\frac{|\nabla w_\varepsilon|^q}{\varepsilon^q}$$

Therefore we deduce

$$-\operatorname{div}(A(x)\nabla w_\varepsilon) + \lambda w_\varepsilon \leq \gamma\frac{|\nabla w_\varepsilon|^q}{\varepsilon^{q-1}} + \varepsilon f(x).$$

Setting $\hat{w}_\varepsilon = \frac{w_\varepsilon}{\varepsilon}$, we have that \hat{w}_ε satisfies

$$-\operatorname{div}(A(x)\nabla \hat{w}_\varepsilon) + \lambda \hat{w}_\varepsilon \leq \gamma|\nabla \hat{w}_\varepsilon|^q + f(x).$$

Since $\hat{w}_\varepsilon \in H_0^1(\Omega) \cap L^\infty(\Omega)$, we can apply Theorem 3.1 hence

$$\hat{w}_\varepsilon \leq \frac{\|f\|_\infty}{\lambda}$$

which means

$$(1 + \varepsilon)u - v \leq \varepsilon \left(\frac{\|f\|_\infty}{\lambda} \right).$$

Letting $\varepsilon \rightarrow 0$ we get $u \leq v$. ■

The arguments of Theorem 3.1 (a priori estimate) and of the next corollary concerning uniqueness are developed in a more general framework in [18], [6] for the case of *unbounded* solutions with $f \notin L^\infty(\Omega)$.

We conclude this section with an existence and uniqueness result.

Theorem 3.2 *Assume (2.2), and that $f \in L^\infty(\Omega)$, and let $\lambda > 0$, $1 \leq q \leq 2$. Then there exists a unique $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ solution of (3.1).*

Proof. Uniqueness is an immediate consequence of Corollary 3.1. Existence may be proved in several ways: we use the approximation argument as in [9]. Define the solutions u_n of

$$\begin{cases} -\operatorname{div}(A(x)\nabla u_n) + \lambda u_n = \gamma \frac{|\nabla u_n|^q}{1 + \frac{1}{n}|\nabla u_n|^q} + f(x) & \text{in } \Omega, \\ u_n \in H_0^1(\Omega) \cap L^\infty(\Omega) \end{cases} \quad (3.9)$$

The solutions u_n exist thanks to the results for the sublinear growth. Using the estimate of Theorem 3.1 we get

$$\|u_n\|_\infty \leq \frac{\|f\|_\infty}{\lambda},$$

hence the sequence u_n is uniformly bounded. It easily follows that u_n is also bounded in $H_0^1(\Omega)$. Next, we only need to prove that u_n is compact in $H_0^1(\Omega)$, since then u_n will converge to some function $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and thanks to the strong convergence we will be able to pass to the limit in the equation and find that u is a solution. So, we are only left with the compactness in $H_0^1(\Omega)$, to this purpose we use the following lemma, which is the basic idea of [9].

Lemma 3.1 *Assume that u_n is a sequence of solutions of*

$$\begin{cases} -\operatorname{div}(A(x)\nabla u_n) + \lambda u_n = H_n(x, \nabla u_n) & \text{in } \Omega, \\ u_n \in H_0^1(\Omega) \cap L^\infty(\Omega) \end{cases}$$

where $\lambda \in \mathbb{R}$ and $H_n(x, \xi)$ is a family of functions satisfying

$$|H_n(x, \xi)| \leq \gamma(1 + |\xi|^2)$$

Let u_n be a bounded sequence in $H_0^1(\Omega) \cap L^\infty(\Omega)$. Then u_n is relatively compact in $H_0^1(\Omega)$.

Proof.

Assume (up to subsequences) that u_n weakly converges to u in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$. Since u_n is uniformly bounded, this also means that u_n converges strongly in $L^p(\Omega)$ for any $p < \infty$.

Take $\psi(u_n - u)$ as test function, where $\psi(s)$ is a real variable function (to be fixed later) such that $\psi(0) = 0$. We get, using the growth of H_n ,

$$\begin{aligned} & \int_{\Omega} A(x) \nabla u_n \cdot \nabla(u_n - u) \psi'(u_n - u) dx + \lambda \int_{\Omega} u_n \psi(u_n - u) dx \\ & \leq \gamma \int_{\Omega} |\nabla u_n|^2 |\psi(u_n - u)| + \gamma \int_{\Omega} \psi(u_n - u) \end{aligned}$$

and then

$$\begin{aligned} & \int_{\Omega} A(x) \nabla(u_n - u) \cdot \nabla(u_n - u) \psi'(u_n - u) dx \leq \\ & 2\gamma \int_{\Omega} |\nabla(u_n - u)|^2 |\psi(u_n - u)| dx + 2\gamma \int_{\Omega} |\nabla u|^2 |\psi(u_n - u)| dx \\ & + \int_{\Omega} (\gamma - \lambda u_n) \psi(u_n - u) dx - \int_{\Omega} A(x) \nabla u \nabla(u_n - u) \psi'(u_n - u) dx - . \end{aligned}$$

Last four terms go to zero using the strong convergence of u_n to u (and the a.e. convergence) and the weak convergence of ∇u_n .

Then using (2.2) we have

$$\alpha \int_{\Omega} |\nabla(u_n - u)|^2 \psi'(u_n - u) dx \leq 2\gamma \int_{\Omega} |\nabla(u_n - u)|^2 |\psi(u_n - u)| dx + \varepsilon_n$$

Choose now $\psi(s)$ such that $\alpha \psi'(s) \geq 2\gamma |\psi(s)| + \frac{\alpha}{2}$ (for example, the function $\psi(s) = se^{bs^2}$ is okay with b large enough). Then you get

$$\frac{\alpha}{2} \int_{\Omega} |\nabla(u_n - u)|^2 dx \leq \varepsilon_n$$

and we conclude that $u_n \rightarrow u$ strongly in $H_0^1(\Omega)$. ■

Remark 3.2 *As far as the existence part is concerned, one can take even $q < 1$ in the previous theorem: the sublinear growth does not bring new problems in terms of a priori estimates and existence. On the other hand, the uniqueness relies on the locally Lipschitz character of the Hamiltonian, this is why in the above result we restricted to $q \geq 1$.*

4 The ergodic limit

Let f belong to $L^\infty(\Omega)$. On account of the results of the previous section, given any $\lambda > 0$ and $q \leq 2$ there exists a (unique, at least if $q \geq 1$) solution to the problem

$$\begin{cases} -\Delta u + \lambda u = |\nabla u|^q + f(x) & \text{in } \Omega, \\ u \in H_0^1(\Omega) \cap L^\infty(\Omega). \end{cases} \quad (4.1)$$

On the other hand, for $\lambda = 0$ and $q > 1$ the problem may have no solution. The limit of u_λ as $\lambda \rightarrow 0$ is called the ergodic limit.¹

In order to study the ergodic limit, an essential tool is played by the following gradient bound. We assume in this section that Ω is a smooth (say, C^2) domain and, for simplicity, that it is connected.

Lemma 4.1 *Let u_λ be the solution of (4.1), where we assume that $q > 1$. Then we have, for every $x \in \Omega$,*

$$|\nabla u_\lambda(x)| \leq \frac{K}{d(x)^{\frac{1}{q-1}}}$$

where K depends only on $\|f\|_{L^\infty(\Omega)}$, q , Ω , and $d(x)$ denotes the distance of x from the boundary.

Proof. See [20], [22]. ■

As a consequence of the above estimate we have the following

¹This name is due to the stochastic interpretation of (4.1), namely on the representation of u_λ in terms of optimal control problems:

$$u_\lambda(x) = \sup_{\mathcal{A}} E_x \left\{ \int_0^{\tau_x} \left[f(X_t) - \frac{q-1}{q^{\frac{q}{q-1}}} |a(X_t)|^{\frac{q}{q-1}} \right] e^{-\lambda t} dt \right\},$$

where X_t solves the SDE

$$dX_t = a(X_t) + \sqrt{2}dW_t, \quad X_0 = x \in \Omega,$$

E_x is the conditional expectation with respect to $X_0 = x$, τ_x is the first exit time from Ω and a belongs to a set \mathcal{A} of admissible control laws. The limit of λu_λ is then related to the properties of ergodicity of the process X_t , see e.g. [3].

Proposition 4.1 *Let u_λ be the unique solution of (4.1), and set*

$$v_\lambda = \|u_\lambda\|_\infty - u_\lambda.$$

Then v_λ is bounded in $W_{loc}^{1,\infty}(\Omega)$.

Proof.

Step 1. For any $\theta \in (0, 1)$, there exists δ_0 , depending only on $\theta, q, \Omega, \|f\|_\infty$ such that, for every $\lambda > 0$,

$$u_\lambda(x) \leq d(x)^\theta + \sup_{\{d(x)=\delta_0\}} u_\lambda^+ \quad \forall x : d(x) \leq \delta_0.$$

Indeed, take $\psi(x) = d(x)^\theta$, with $\theta \in (0, 1)$ and δ_0 sufficiently small so that $d(x)$ is smooth when $d(x) < \delta_0$. Then

$$\begin{aligned} & -\Delta\psi + \lambda\psi - |\nabla\psi|^q - f(x) = \\ & \theta(1-\theta)d^{\theta-2} - \theta d^{\theta-1}\Delta d + \lambda d^\theta - \theta^q d^{(\theta-1)q} - f(x) \\ & \geq \theta d^{\theta-2} \left[(1-\theta) - d\Delta d - \theta^{q-1} d^{2-q+\theta(q-1)} \right] - \|f\|_\infty \end{aligned}$$

Since $1 < q \leq 2$, we have that ψ is a supersolution in the subset $\{x \in \Omega : d(x) < \delta_0\}$, for some $\delta_0 > 0$ depending only on θ, q, Ω, f . Since $\psi + \sup_{\{d(x)=\delta_0\}} u_\lambda^+$

is still a supersolution, we conclude by comparison.

Step 2. First observe that u_λ is bounded from below; indeed, we have $-\Delta u_\lambda^- \leq |f(x)|$, hence $\|u_\lambda^-\|_\infty \leq c \|f\|_\infty$. Therefore, we deduce from Step 1 that, for some constant C_0 ,

$$\|u_\lambda\|_\infty \leq C_0 + \sup_{\{d(x) \geq \delta_0\}} u_\lambda$$

hence there exists x_λ such that $d(x_\lambda) \geq \delta_0$ such that

$$0 \leq v_\lambda(x) \leq C_0 + u_\lambda(x_\lambda) - u_\lambda(x)$$

Using Lemma 4.1 we deduce that v_λ is locally uniformly bounded. Since $\nabla v_\lambda = -\nabla u_\lambda$, again from Lemma 4.1 we deduce that $|\nabla v_\lambda|$ is locally uniformly bounded too, hence we conclude. \blacksquare

We are ready to prove that the behaviour of u_λ is uniquely determined by the existence of solutions of the limiting problem.

Theorem 4.1 *Let $1 < q \leq 2$, and $f \in L^\infty(\Omega)$. Let u_λ be the solution of (4.1). Then we have*

(i) *If there exists a solution φ of*

$$\begin{cases} -\Delta\varphi = |\nabla\varphi|^q + f(x) & \text{in } \Omega, \\ \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega), \end{cases} \quad (4.2)$$

then $u_\lambda \rightarrow \varphi$ in $H_0^1(\Omega)$ as $\lambda \rightarrow \infty$.

(ii) *If problem (4.2) has no solution, then $u_\lambda(x) \rightarrow +\infty$ for every $x \in \Omega$, and if we set $v_\lambda = \|u_\lambda\|_\infty - u_\lambda$ then*

$$v_\lambda \rightarrow v_0, \quad \lambda u_\lambda \rightarrow c_0$$

locally uniformly, where c_0 is the unique constant such that the problem

$$\begin{cases} -\Delta v + |\nabla v|^q = c_0 - f(x) & \text{in } \Omega, \\ \lim_{x \rightarrow \partial\Omega} v(x) = \infty, \end{cases} \quad (4.3)$$

admits a (classical) solution and v_0 is the unique solution such that $\min_{\Omega} v_0(x) = 0$.

Proof.

Proof of (i). Assume by contradiction that (for a subsequence of λ , not relabeled), we have $\|u_\lambda\|_\infty \rightarrow \infty$. Define $v_\lambda = \|u_\lambda\|_\infty - u_\lambda$, hence v_λ solves

$$-\Delta v_\lambda + \lambda v_\lambda + |\nabla v_\lambda|^q = -f(x) + \lambda \|u_\lambda\|_\infty. \quad (4.4)$$

Consider $\hat{v}_\lambda = (1 + \lambda)v_\lambda$, then

$$-\Delta \hat{v}_\lambda + \lambda \hat{v}_\lambda + (1 + \lambda)^{1-q} |\nabla \hat{v}_\lambda|^q = -(1 + \lambda)f(x) + (1 + \lambda)\lambda \|u_\lambda\|_\infty$$

Let φ be solution of (4.2) and set $w_\lambda = \hat{v}_\lambda + \varphi - k$, where $k > 0$. We obtain that

$$-\Delta w_\lambda + \lambda w_\lambda + (1 + \lambda)^{1-q} |\nabla \hat{v}_\lambda|^q - |\nabla \varphi|^q = \lambda(\varphi(x) - k - f(x) + (1 + \lambda)\|u_\lambda\|_\infty)$$

By convexity we have

$$|\nabla \hat{v}_\lambda|^q \leq \frac{1}{1 + \lambda} |(1 + \lambda)\nabla \varphi|^q + \frac{\lambda}{1 + \lambda} \left| \frac{1 + \lambda}{\lambda} \nabla(\hat{v}_\lambda + \varphi) \right|^q$$

hence

$$(1 + \lambda)^{1-q} |\nabla \hat{v}_\lambda|^q \leq |\nabla \varphi|^q + \lambda^{1-q} |\nabla(\hat{v}_\lambda + \varphi)|^q.$$

We deduce that w_λ satisfies

$$-\Delta w_\lambda + \lambda w_\lambda + \lambda^{1-q} |\nabla w_\lambda|^q \geq \lambda(\varphi(x) - k - f(x) + (1 + \lambda) \|u_\lambda\|_\infty).$$

Assuming that $\|u_\lambda\|_\infty \rightarrow \infty$ implies that, for λ small,

$$-\Delta w_\lambda + \lambda w_\lambda + \lambda^{1-q} |\nabla w_\lambda|^q \geq 0,$$

and moreover, since $w_\lambda = (1 + \lambda) \|u_\lambda\|_\infty - k$ on $\partial\Omega$, we have that $w_\lambda \geq 0$ on the boundary (even more, w_λ^- has compact support in Ω). Since $w_\lambda^- \in H_0^1(\Omega) \cap L^\infty(\Omega)$, we deduce by the weak maximum principle (see Theorem 3.1) that $w_\lambda \geq 0$, hence that

$$v_\lambda(1 + \lambda) + \varphi - k \geq 0 \quad \text{in } \Omega.$$

Recall by Proposition 4.1 that v_λ is bounded in $W_{loc}^{1,\infty}(\Omega)$. Fix a compact set $K \subset \Omega$, and let $v_0 \in C(K)$ be a limit of v_λ in K as $\lambda \rightarrow 0$. We deduce that

$$v_0 \geq k - \varphi \quad \text{in } K.$$

Since k is arbitrary and v_0 is bounded we get a contradiction.

This proves that $\|u_\lambda\|_\infty$ remains bounded. Then we can deduce, using e.g. Lemma 3.1, that u_λ is relatively compact in $H_0^1(\Omega)$. Then u_λ converges (up to subsequences) to a solution of (4.2) (which is actually locally Lipschitz, due to estimate of lemma 4.1, hence a classical solution). This solution is unique because of the (classical) strong maximum principle, hence we conclude that the whole sequence u_λ converges.

Proof of (ii). First observe that, as $\lambda \rightarrow 0$, we have that $\|u_\lambda\|_\infty \rightarrow \infty$; indeed, should $\|u_\lambda\|_\infty$ remain bounded, then u_λ will converge (up to a subsequence) to a solution of (4.2), a fact which contradicts our assumption. Therefore, we have $\|u_\lambda\|_\infty \rightarrow \infty$.

On the other hand, by Theorem 3.1 we have $\lambda \|u_\lambda\|_\infty \leq \|f\|_\infty$. Therefore, we have

$$-\Delta v_\lambda + \lambda v_\lambda + |\nabla v_\lambda|^q = -f(x) + \lambda \|u_\lambda\|_\infty \geq -2\|f\|_\infty.$$

Now let $q < 2$; in the domain $\{x \in \Omega : d(x) < \delta_0\}$, consider the function $\psi = \sigma \left(d(x) + \frac{1}{n}\right)^{-\alpha} - M$, where

$$\alpha = \frac{2-q}{q-1}, \quad M = \sigma \left(\delta_0 + \frac{1}{n}\right)^{-\alpha} + \sup_{d(x)=\delta_0} v_\lambda$$

and δ_0 is to be chosen (sufficiently small so that $d(x)$ is smooth in this domain). Computing we have

$$\begin{aligned} -\Delta\psi + \lambda\psi + |\nabla\psi|^q &= \alpha \sigma \left(d(x) + \frac{1}{n}\right)^{-\alpha-2} \left[-(\alpha+1) + \left(d(x) + \frac{1}{n}\right) \Delta d \right. \\ &\quad \left. + (\alpha\sigma)^{q-1} \left(d(x) + \frac{1}{n}\right)^{2+\alpha-(\alpha+1)q} + \frac{\lambda}{\alpha} \left(d(x) + \frac{1}{n}\right)^2 \right] - \lambda M \end{aligned}$$

where we used that $|\nabla d(x)| = 1$. The value of $\alpha = \frac{2-q}{q-1}$ implies that $2 + \alpha = (\alpha + 1)q$ hence we get

$$\begin{aligned} -\Delta\psi + \lambda\psi + |\nabla\psi|^q &= \alpha \sigma \left(d(x) + \frac{1}{n}\right)^{-\alpha-2} \left[-(\alpha+1) + \left(d(x) + \frac{1}{n}\right) \Delta d \right. \\ &\quad \left. + (\alpha\sigma)^{q-1} + \frac{\lambda}{\alpha} \left(d(x) + \frac{1}{n}\right)^2 \right] - \lambda M \end{aligned}$$

Choosing σ such that $(\alpha\sigma)^{q-1} < \alpha + 1$, then δ_0 and n sufficiently small, we obtain that

$$-\Delta\psi + \lambda\psi + |\nabla\psi|^q \leq -2\|f\|_\infty$$

in $\{d(x) < \delta_0\}$. The value of M implies that $\psi \leq v_\lambda$ on $\{d(x) = \delta_0\}$, and, if λ is small, we have $v_\lambda \geq \psi$ on $\partial\Omega$ as well. We conclude that

$$v_\lambda \geq \psi \quad \text{in } \{d(x) < \delta_0\}.$$

Observe that M depends on λ (and n) but is uniformly bounded, since v_λ is locally uniformly bounded, hence there exists some constant K such that

$$v_\lambda \geq \sigma d(x)^{-\alpha} - K \quad \text{in } \{d(x) < \delta_0\}.$$

Now, by Proposition 4.1, there exists a subsequence of λ (not relabeled) and a function $v_0 \in W_{loc}^{1,\infty}(\Omega)$ such that $v_\lambda \rightarrow v_0$ locally uniformly in Ω . We deduce that

$$v_0 \geq \sigma d(x)^{-\alpha} - K \quad \text{in } \{d(x) < \delta_0\},$$

which implies that $v_0 \rightarrow +\infty$ as $x \rightarrow \partial\Omega$. When $q = 2$, a similar estimate can be obtained using for ψ a logarithm function.

Moreover, by elliptic regularity, the convergence of v_λ to v_0 locally holds in $C^2(\Omega)$, which allows one to pass to the limit in the equation (4.4) satisfied by v_λ . Finally, still up to subsequences, we have that

$$\lambda \|u_\lambda\|_\infty \rightarrow c_0$$

for some constant c_0 , and we conclude that v_0 satisfies (4.3). Note also that λu_λ itself converges to c_0 locally uniformly, since $\lambda u_\lambda = \lambda \|u_\lambda\|_\infty - \lambda v_\lambda$ and $\lambda v_\lambda \rightarrow 0$ because v_λ is locally bounded.

To conclude, we use the (fundamental) result in [20] which says that c_0 is unique (i.e. the unique constant such that (4.3) may have solution) and that problem (4.3) has a unique (classical) solution up to addition of a constant. In particular, we deduce that v_0 is the unique solution such that $\min_{\Omega} v_0 = 0$. The uniqueness of c_0 and v_0 implies that the whole sequences v_λ and λu_λ converge to v_0 and to c_0 respectively. ■

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