Homogenization of a degenerate parabolic problem in a highly heterogeneous medium with highly anisotropic fibers

A. Boughammoura

University of Monastir, Tunisia

CIMPA 2009
We consider the homogenization of a heat transfer problem in a periodic medium consisting of a set of highly anisotropic fibers surrounded by insulating layers, the whole is being embedded in a third material having a conductivity of order 1. The conductivity along the fibers is of order 1 but the conductivities in the transverse directions and in the insulating layers are very small, and related to the scales $\mu$ and $\lambda$ respectively. We assume that $\mu$ (resp. $\lambda$) tends to zero with a rate $\mu = \mu(\varepsilon)$ (resp. $\lambda = \lambda(\varepsilon)$), where $\varepsilon$ is the size of the basic periodicity cell. The heat capacities $c_i$ of the $i$-th component are positive, but may vanish at some subsets such that the problem can be degenerate (parabolic-elliptic). We show that the critical values of the problem are $\gamma = \lim_{\varepsilon \to 0} \frac{\varepsilon^2}{\mu}$ and $\delta = \lim_{\varepsilon \to 0} \frac{\varepsilon^2}{\lambda}$, and we identify the homogenized limit depending on whether $\gamma$ and $\delta$ are zero, strictly positive, finite or infinite.
THE PHYSICAL and GEOMETRY ASSUMPTION
Now, we introduce a reference periodic medium as follows. We denote by \( \tilde{Y} \) and \( Y \) the unit cube in \( \mathbb{R}^{N-1} \) and \( \mathbb{R}^N \) respectively: \( Y = \tilde{Y} \times I, \ I = ]0,1[ \). We assume that \( \tilde{Y} \) is partitioned as \( \tilde{Y} = \tilde{Y}_1 \cup \tilde{Y}_{13} \cup \tilde{Y}_2 \cup \tilde{Y}_{23} \cup \tilde{Y}_3 \) where \( \tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3 \) are three connected open subsets such that \( \tilde{Y}_1 \cap \tilde{Y}_2 = \emptyset, \partial \tilde{Y} \cap \tilde{Y}_3 = \emptyset \) and where \( \tilde{Y}_{\alpha3}, \ \alpha = 1,2 \) is the interface between \( \tilde{Y}_\alpha \) and \( \tilde{Y}_3 \); thus \( \tilde{Y}_3 \) separates \( \tilde{Y}_1 \) and \( \tilde{Y}_2 \) (see Figure 1).
Figure: A typical basic cell $\tilde{Y}$ in 2D
For $i = 1, 2, 3$ we denote $\chi_i$ the characteristic function of $Y_i := \tilde{Y}_i \times I$ and $\theta_1, \theta_2, \theta_3$ their respective Lebesgue measures are supposed to be of the same magnitude order. Let $\tilde{E}_i$ the $\mathbb{Z}^{N-1}$-translates of $\tilde{Y}_i$, i.e., $\tilde{E}_i := \tilde{Y}_i + \mathbb{Z}^{N-1}$ and $\tilde{\Gamma}_{\alpha 3}$, $\alpha = 1, 2$ the surface separating $E_\alpha$ and $E_3$. We shall assume that only $E_2$ is connected. We introduce the contracted sets $\tilde{Y}_i^\varepsilon := \varepsilon \tilde{Y}_i$, $\tilde{E}_i^\varepsilon := \varepsilon \tilde{E}_i$, $i = 1, 2, 3$ and $\tilde{\Gamma}_{\alpha 3}^\varepsilon := \varepsilon \tilde{\Gamma}_{\alpha 3}$, $\alpha = 1, 2$, where $\varepsilon$ is a small positive parameter. We assume that $\tilde{E}_1^\varepsilon$ is filled with a material having a finite conductivity in the direction $x_N$ of the fibers and a conductivity tending to zero with a rate $\mu(\varepsilon)$ in the transverse directions $x_\alpha$, $\alpha = 1, 2, \ldots, N - 1$. The phase $\tilde{E}_3^\varepsilon$ consists of a material with a conductivity that tends to zero with a rate $\lambda(\varepsilon)$, and the matrix $\tilde{E}_2^\varepsilon$ is filled by a material with a finite conductivity.
Now, let $\tilde{\Omega}$ be a regular bounded domain in $\mathbb{R}^{N-1}$. We denote by $\tilde{\Omega}_i^\varepsilon := \tilde{\Omega} \cap \tilde{E}_i^\varepsilon$, and $\tilde{S}_{\alpha 3}^\varepsilon := \tilde{\Omega} \cap \tilde{\Gamma}_{\alpha 3}$. Finally, let $\Omega := \tilde{\Omega} \times I$ be the cylinder having a base $\tilde{\Omega}$ and a height 1 and $\Omega_i^\varepsilon := \tilde{\Omega}_i^\varepsilon \times I$, $i = 1, 2, 3$.

Henceforth, $x = (\tilde{x}, x_N)$ and $y = (\tilde{y}, y_N)$ denote points of $\mathbb{R}^N$ and $Y$ respectively and by $\tilde{y}$ and $\tilde{x}$ we denote the transverse vectors $(y_1, \cdots, y_{N-1})$ and $(x_1, \cdots, x_{N-1})$ respectively. We use the notation $\partial_{x_i}$ for the partial derivative with respect to $x_i$. Let $T > 0$ be given. We define, then, the corresponding space-time domains $Q = (0, T) \times \Omega$ and $Q_i^\varepsilon = (0, T) \times \Omega_i^\varepsilon$, $i = 1, 2, 3$. 
For $k = 1, 2, 3$, let $c_k, a_k \in L^\infty(\mathbb{R}^N)$. Here $c_k$ is the heat capacity of the $k$-th component. These functions are $Y$-periodic with respect to $y$ with a period $Y$ and verify the following assumptions

(A1) : $0 \leq c_k(y)$ a.e $y$

(A2) : $a_0 \leq a_k(y)$ a.e $y$ where $a_0 > 0$, independent of $y$.

(A3) : $a_1$ is independent of the vertical coordinate, i.e., $a_1(y) := a_1(\widetilde{y})$. The corresponding $\varepsilon$-periodic coefficients are defined by

$$c_\varepsilon^k(x) = c_k(\frac{x}{\varepsilon}), \quad a_\varepsilon^k(x) = a_k(\frac{x}{\varepsilon}), \quad x \in \Omega^\varepsilon_k, \quad k = 1, 2, 3. \quad (1)$$

Let:

$$A_1^\varepsilon(x) = a_1^\varepsilon(x) \begin{pmatrix} \mu(\varepsilon) I_{N-1} & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2^\varepsilon(x) = a_2^\varepsilon(x) I_N, \quad A_3^\varepsilon(x) = a_3^\varepsilon(x) I_N,$$

where $I_N$ is the matrix of identity.
Then, the conductivity tensors of the material filling the sets $E_i^\varepsilon$ are respectively $A_{1}^\varepsilon$, $A_{2}^\varepsilon$, $\lambda(\varepsilon)A_{3}^\varepsilon$ and the global conductivity and the heat capacity of the medium are respectively

$$A^\varepsilon(x) = \chi_1^\varepsilon(x)A_1^\varepsilon(x) + \chi_2^\varepsilon(x)A_2^\varepsilon(x) + \lambda \chi_3^\varepsilon(x)A_3^\varepsilon(x)$$

and

$$c^\varepsilon(x) = \chi_1^\varepsilon(x)c_1^\varepsilon(x) + \chi_2^\varepsilon(x)c_2^\varepsilon(x) + \chi_3^\varepsilon(x)c_3^\varepsilon(x).$$

where

$$\chi_k^\varepsilon(x) = \chi_k(\frac{x}{\varepsilon}), \ k = 1, 2, 3.$$
Let us assume that the lateral and bottom boundaries of Ω are maintained at a fixed temperature (homogeneous Dirichlet condition), while the top boundary is insulated (homogeneous Neumann condition), and that the initial distribution of the temperature on Ω is given for every $\varepsilon$ as

$$u_0^\varepsilon(x) = \sum_{i=1}^{3} \chi^\varepsilon_i(x) u_0^\varepsilon_i(x).$$

Then, the evolution of the temperature $u^\varepsilon(t, x)$ is governed by the following initial boundary value problem, being in fact, a sequence of problems $\mathcal{P}_\varepsilon$ indexed by $\varepsilon$:
\begin{equation}
\frac{\partial}{\partial t}(c^\varepsilon(x)u^\varepsilon(t, x)) = \text{div}(A^\varepsilon(x) \nabla u^\varepsilon(t, x)) + f^\varepsilon(t, x), \quad (t, x) \in \mathbb{R}_+^* \times \Omega
\end{equation}

\begin{align*}
\frac{\partial}{\partial x_N} u^\varepsilon(t, \tilde{x}, 1) &= 0, \quad \tilde{x} \in \tilde{\Omega}, \quad t > 0 \\
u^\varepsilon(0, x) &= u_0^\varepsilon(x), \quad x \in \Omega \\
u^\varepsilon(t, \tilde{x}, x_N) &= 0, \quad (\tilde{x}, x_N) \in \partial \tilde{\Omega} \times [0, 1[, \quad t > 0.
\end{align*}

where \( f^\varepsilon(t, x) = f(t, x) \), \( f \in L^2(Q) \) represents a given time-dependent heat source. The coefficients \( c^\varepsilon \) and \( a^\varepsilon \) being discontinuous, the above equation has to be interpreted in the sense of distributions on \( \Omega \).
To have a weak formulation of the problem which allows us to handle less regular data and especially give a precise meaning to the initial and the boundary conditions, we shall use the convenient mathematical model for the problem $\mathcal{P}_\varepsilon$, built in [Mabrouk, 2005] using the functional framework, developed by [Showalter, 1999] for degenerate parabolic equations. Let us recall the precise meaning of the initial condition.
We define $H_{LB}^1(\Omega)$ as the closed subspace of $H^1(\Omega)$:

$$H_{LB}^1(\Omega) = \left\{ u \in H^1(\Omega) : u = 0 \text{ on } \partial\tilde{\Omega} \times [0, 1[ \right\}$$

The subscript $L$ (resp. $B$) stands for the lateral (resp. bottom) boundary.

Let $V = H_{LB}^1(\Omega)$, $\mathcal{V} = L^2(0, T; V)$ and $V' = (H_{LB}^1(\Omega))^\prime$, $\mathcal{V}' = L^2(0, T; V')$ be their dual spaces. For every $\varepsilon > 0$, let $B^\varepsilon, A^\varepsilon : V \rightarrow V'$, the continuous operators are defined by the two continuous bilinear forms on $V \times V$:

$$(u, v) \in V \times V \mapsto \langle B^\varepsilon u, v \rangle_{V', V} = b^\varepsilon(u, v) = \int_\Omega c^\varepsilon(x)u(x)v(x)dx$$

$$(u, v) \in V \times V \mapsto \langle A^\varepsilon u, v \rangle_{V', V} = a^\varepsilon(u, v) = \int_\Omega A^\varepsilon(x)\nabla u(x)\nabla v(x)dx$$
Let $V^\varepsilon_b$ be the completion of $V$ with the semi-scalar product, defined by the form $b^\varepsilon$ and let $V'^\varepsilon_b$ be its dual. Then, we have $V^\varepsilon_b = \{ u : (c^\varepsilon)^{1/2} u \in L^2(\Omega) \}$ and $V'^\varepsilon_b = \{ (c^\varepsilon)^{1/2} u, \ u \in L^2(\Omega) \}$. The operator $B^\varepsilon$ admits a continuous extension from $V^\varepsilon_b$ into $V'^\varepsilon_b$ denoted also by $B^\varepsilon$. Given $f^\varepsilon \in L^2(0, T; L^2(\Omega))$ or more generally $f^\varepsilon$ in $V'$ and $w^\varepsilon_0$ in $V'^\varepsilon_b$, we are now able to give a weak formulation of the above initial-boundary value problem as the following abstract Cauchy problem:

Find $u \in V : \ \frac{d}{dt} B^\varepsilon u + A^\varepsilon u = f^\varepsilon \in V', \ \ B^\varepsilon u(0) = w^\varepsilon_0 \in V'^\varepsilon_b$

Here, $A^\varepsilon$ and $B^\varepsilon$ are the realization of $A^\varepsilon$ and $B^\varepsilon$ as operators from $V$ to $V'$, that is precisely $(A^\varepsilon u(t), B^\varepsilon u(t)) = (A^\varepsilon(u(t)), B^\varepsilon(u(t)))$ for a.e. $t \in (0, T)$. Let us underline that, in the abstract formulation above, we implicitly require that $\frac{d}{dt} B^\varepsilon u$ belongs to $V'$. This allows us to give a precise meaning to the initial condition $B^\varepsilon u(0)$. 
Indeed, the scalar products in $V_b^\varepsilon, V'_b^\varepsilon$ are defined by

$$(B^\varepsilon u, B^\varepsilon v)_{V'_b^\varepsilon} = \langle B^\varepsilon u, v \rangle_{V'_b^\varepsilon, V_b^\varepsilon} = (u, v)_{V_b^\varepsilon} = \int_\Omega c^\varepsilon(x)u(x)v(x)dx$$

Thus, given $u_0^\varepsilon$ in $V_b^\varepsilon$ and $w_0^\varepsilon$ in $V'_b^\varepsilon$ related by $w_0^\varepsilon = c^\varepsilon u_0^\varepsilon$, we can express the initial condition by one of the two equivalent equalities

$$(B^\varepsilon u^\varepsilon)(0) = B^\varepsilon u^\varepsilon(0) = w_0^\varepsilon \in V'_b^\varepsilon \iff (c^\varepsilon)^{\frac{1}{2}} u^\varepsilon(0) = (c^\varepsilon)^{\frac{1}{2}} u_0^\varepsilon \in L^2(\Omega)$$

We define the Hilbert space $W_2^\varepsilon(0, T) := \{ u \in \mathcal{V} : \frac{d}{dt}B^\varepsilon \in \mathcal{V}' \}$, then, the abstract Cauchy problem can, thereby, be written more explicitly as:
Find \( u \) in \( \mathcal{W}_{2}^{\varepsilon}(0, T) \) s.t.

\[
\begin{aligned}
\frac{d}{dt} \mathcal{B}^{\varepsilon} u(t) + \mathcal{A}^{\varepsilon} u(t) &= f^{\varepsilon}(t) \in V' \quad \text{for a.e. } t \in (0, T), \\
\mathcal{B}^{\varepsilon} u(0) &= w_{0}^{\varepsilon} \text{ in } V_{b}^{\varepsilon}
\end{aligned}
\]

The initial condition is meaningful since \( u \) is in \( \mathcal{W}_{2}^{\varepsilon}(0, T) \). Some equivalent variational formulations of the problem \( PC_{\varepsilon} \) are given in the Proposition 1.2 of [?]. For the present study, we need only the following useful variational formulation. The function \( u \) is a solution of the problem \( PC_{\varepsilon} \), if and only if, \( u \) is in \( \mathcal{V} \) and for all \( v \) in \( H^{1}(0, T; \mathcal{V}) \) with \( v(T) = 0 \), we have

\[
\begin{align*}
- \int_{Q} c^{\varepsilon} u^{\varepsilon} v' \, dxdt + \mu \int_{Q_{1}^{\varepsilon}} a_{1}^{\varepsilon}(x) \nabla_{\tilde{x}} u^{\varepsilon} \nabla_{\tilde{x}} v \, dxdt + \int_{Q_{1}^{\varepsilon}} a_{1}^{\varepsilon}(x) \partial_{x_{N}} u^{\varepsilon} \partial_{x_{N}} v \, dxdt \\
+ \int_{Q_{2}^{\varepsilon}} a_{2}^{\varepsilon}(x) \nabla_{x} u^{\varepsilon} \nabla_{x} v \, dxdt + \lambda \int_{Q_{3}^{\varepsilon}} a_{3}^{\varepsilon}(x) \nabla_{x} u^{\varepsilon} \nabla_{x} v \, dxdt \\
= \int_{Q} f v \, dxdt + \int_{\Omega} c^{\varepsilon} u_{0}^{\varepsilon} v(0, x) \, dx.
\end{align*}
\]
Throughout this work, we shall assume that:

\[ u_0^\varepsilon(x) = u_0(x) \in L^2(\Omega). \]

Thus \( u_0^\varepsilon \) belongs to \( V_\varepsilon^b \) and the initial condition is meaningful for any \( \varepsilon > 0 \).

Our objective is to study the behavior of the sequence \( \{u^\varepsilon\} \) as \( \varepsilon \to 0 \). This will be achieved below. We show that the limit depends on the values \( \gamma = \lim_{\varepsilon \downarrow 0} \frac{\varepsilon^2}{\mu} \) and \( \delta = \lim_{\varepsilon \downarrow 0} \frac{\varepsilon^2}{\lambda} \). We consider, in Section 3, the case \( \delta < \infty \) which will be processed according to whether \( \gamma \) is zero, strictly positive, finite or infinite. The critical case \( 0 < \delta < \infty \), which gives the most interesting results, will be presented with sufficient details. The case \( \delta = \infty \) is shortly considered in Section 4, since the results of the convergence are exactly the same as in [Mabrouk, Boughammoura, 2004, 2007], to which we refer for the details that remain unchanged. Our further analysis will be based on the method of the two-scale convergence [Allaire, 19992, Nguetseng, 1989].
The problems of homogenization in composite media with fibers was considered by many authors in several contexts, see [Bouchité & Bellieud, 1998, 2002; Cailleria & Dinari, 1987; Cherednichenko, Smyshlyaev & Zhikov, 2006; Panasenko, 1991; Sili, 2004] and further references therein. Most of the previous works dealt with the case of the fiber-reinforced composite material. Motivated by the study of the effects of the combination of high heterogeneity and high anisotropy in the overall behavior of composite media (see [Mabrouk & Boughammoura, 2004, 2007]), we consider here, a special class of fiber structure exhibiting a non-standard homogenized problem. Especially, the present work concerns a degenerate elliptic-parabolic problem in a highly heterogeneous medium involving highly anisotropic fibers with high contrasts between the conductivity along the fibers and the conductivities in the transverse directions, for which, in the critical case, the homogenized problem is an integro-differential one, displaying non-locality along the fibers.
Fewer are the papers dealing with highly anisotropic "coated" fibers. As far as we know, the closest work in this context seems to be the recent paper [Cherednichenko, Smyshlyaev & Zhikov,2006], where the authors studied the elliptic stationary case in a composite, with highly anisotropic fibers with the so called "double porosity" type scaling (i.e. \( \mu = \varepsilon^2 \)), expressing the high contrast between the conductivity along the fibers and the conductivity in the transverse directions. Thus, our work goes beyond that of [Cherednichenko, Smyshlyaev & Zhikov,2006] at least in three directions:
1. We consider an evolution (and degenerate) problem.
2. We investigate a more complex and a more interesting transversal geometry.
3. We analyse a fiber-reinforced composite taking into account the combined effects of fiber coatings together with the highly anisotropy of the material in the fibers.
Our results reflect these facts and a much more precise comparison with those results is presented below in remark 3.3.
Lemma

There exists a constant $C$ such that, for every $v \in H_{LB}^1(\Omega)$, we have

$$
\|v\|_{L^2(\Omega)}^2 \leq C \left( \|\partial_{x_N} v\|_{L^2(\Omega_1^\varepsilon)}^2 + \|\nabla v\|_{L^2(\Omega_2^\varepsilon)}^2 + \varepsilon^2 \|\nabla v\|_{L^2(\Omega_3^\varepsilon)}^2 \right)
$$

A priori estimates
A priori estimates

Let $\delta = \lim_{\varepsilon \downarrow 0} \frac{\varepsilon^2}{\lambda}$. Then
A priori estimates

Let \( \delta = \lim_{\varepsilon \downarrow 0} \frac{\varepsilon^2}{\lambda} \). Then

- **Case 1:** \( \delta < +\infty \), \( f^\varepsilon = f \). There exists a constant \( C \) such that:

\[
\left( \| u^\varepsilon \|_{L^2(Q)} , \| u^\varepsilon \|_{L^\infty(0,T,V^\varepsilon_b)} \right) \leq C
\]

\[
\left( \mu \| \nabla u^\varepsilon \|_{L^2(Q^\varepsilon_1)}^2 , \| \partial_{x_N} u^\varepsilon \|_{L^2(Q^\varepsilon_1)} \right) \leq C
\]

\[
\| \nabla u^\varepsilon \|_{L^2(Q^\varepsilon_2)} \leq C
\]

\[
\lambda \| \nabla u^\varepsilon \|_{L^2(Q^\varepsilon_3)}^2 \leq C
\]
A priori estimates

Let \( \delta = \lim_{\varepsilon \downarrow 0} \varepsilon^2 \frac{\varepsilon^2}{\lambda} \). Then

- **Case 2:** \( \delta = +\infty \). We introduce the scaled sequences:

  \[
  \nu^\varepsilon = \sqrt{\lambda} \frac{\varepsilon^2}{\varepsilon} u^\varepsilon \quad \text{and} \quad z^\varepsilon = \sqrt{\lambda} \frac{\varepsilon^2}{\varepsilon} v^\varepsilon .
  \]

  Then
A priori estimates

Let $\delta = \lim_{\varepsilon \downarrow 0} \frac{\varepsilon^2}{\lambda}$. Then

- **Case 2:** $\delta = +\infty$. We introduce the scaled sequences:
  
  \[ v^\varepsilon = \frac{\sqrt{\lambda}}{\varepsilon} u^\varepsilon \quad \text{and} \quad z^\varepsilon = \frac{\sqrt{\lambda}}{\varepsilon} v^\varepsilon. \]

  Then

- **Case 2.1:** $f^\varepsilon = \left( \chi^\varepsilon_1 + \chi^\varepsilon_2 + \frac{\sqrt{\lambda}}{\varepsilon} \chi^\varepsilon_3 \right) f$. Then the above estimates are true and we have

  \[
  \| u^\varepsilon \|_{L^2(\Omega^\varepsilon_1 \cup \Omega^\varepsilon_2)} \leq C, \quad \| u^\varepsilon \|_{L^2(\Omega^\varepsilon_3)} \leq C + C \frac{\varepsilon}{\sqrt{\lambda}}.
  \]

  \[
  \left( \mu \| \nabla_{\tilde{x}} v^\varepsilon \|_{L^2(Q^\varepsilon_1)}^2, \| \partial_{x_N} v^\varepsilon \|_{L^2(Q^\varepsilon_1)}^2 \right) \leq C
  \]

  \[
  \| \nabla v^\varepsilon \|_{L^2(Q^\varepsilon_2)} \leq C
  \]

  \[
  \lambda \| \nabla v^\varepsilon \|_{L^2(Q^\varepsilon_3)}^2 \leq C.
  \]

Hence,

\[
\| v^\varepsilon \|_{L^2(\Omega^\varepsilon_1 \cup \Omega^\varepsilon_2)} \leq C + C \frac{\varepsilon}{\sqrt{\lambda}}, \quad \| z^\varepsilon \|_{L^2(\Omega^\varepsilon_3)} \leq C \frac{\varepsilon}{\sqrt{\lambda}}.
\]
A priori estimates

Let $\delta = \lim_{\varepsilon \downarrow 0} \frac{\varepsilon^2}{\lambda}$. Then

- **Case 2:** $\delta = +\infty$. We introduce the scaled sequences:

  $v^\varepsilon = \sqrt{\lambda} u^\varepsilon$ and $z^\varepsilon = \frac{\varepsilon}{\sqrt{\lambda}} v^\varepsilon$. Then

- **Case 2.2:** $f^\varepsilon = f$.

\[
\left( \mu \left\| \nabla_x v^\varepsilon \right\|^2_{L^2(Q^\varepsilon_1)}, \left\| \partial_{x_N} v^\varepsilon \right\|_{L^2(Q^\varepsilon_1)}, \left\| \nabla_x v^\varepsilon \right\|^2_{L^2(Q^\varepsilon_2)}, \lambda \left\| \nabla_x v^\varepsilon \right\|^2_{L^2(Q^\varepsilon_3)} \right) \leq C
\]

\[
\left\| v^\varepsilon \right\|^2_{L^2(Q^\varepsilon_1 \cup Q^\varepsilon_2)} \leq C, \quad \left\| v^\varepsilon \right\|^2_{L^2(Q^\varepsilon_3)} \leq C + C \frac{\varepsilon^2}{\lambda}
\]

\[
\left( \mu \left\| \nabla_x z^\varepsilon \right\|^2_{L^2(Q^\varepsilon_1)}, \left\| \partial_{x_N} z^\varepsilon \right\|_{L^2(Q^\varepsilon_1)}, \left\| \nabla_x z^\varepsilon \right\|^2_{L^2(Q^\varepsilon_2)}, \lambda \left\| \nabla_x z^\varepsilon \right\|^2_{L^2(Q^\varepsilon_3)} \right) \leq C \frac{\varepsilon^2}{\lambda}
\]

\[
\left\| z^\varepsilon \right\|^2_{L^2(Q^\varepsilon_1 \cup Q^\varepsilon_2)} \leq C \frac{\varepsilon^2}{\lambda}, \quad \left\| z^\varepsilon \right\|^2_{L^2(Q^\varepsilon_3)} \leq C
\]
Convergence
Definition of the two-scale convergence

Let $p > 1$ and $p'$ its conjugate.
A sequence $u^\varepsilon$ in $L^p(Q)$ two-scale converges to a function $u^0 \in L^p(Q \times Y)$, and we denote this $u^\varepsilon \rightharpoonup^{2s,p} u^0$ ($u^\varepsilon \rightharpoonup^{2s} u^0$ if $p=2$), if, for any $\phi(t, x, y) \in \mathcal{D}(Q, C_\#(Y))$,

$$\lim_{\varepsilon \to 0} \int_Q u^\varepsilon(t, x)\phi(t, x, \frac{x}{\varepsilon}) \, dt \, dx = \int_Q \int_Y u^0(t, x, y)\phi(t, x, y) \, dt \, dx \, dy. \quad (2s)$$

Convergence (2s) can be extended to test functions $\phi \in L^{p'}(Q, C_\#(Y))$ or $\phi \in C(Q, L^{p'}_\#(Y))$. 
Lemma

Let $\gamma := \lim_{\varepsilon \to 0} \frac{\varepsilon^2}{\mu}$. Assume that $\delta < +\infty$, $\gamma < +\infty$ and $f^\varepsilon = f$.

There exists $u_2 \in L^2 \left( 0, T; H^1_{LB} (\Omega) \right)$, $v_1 \in L^2 \left( Q; H^1_\# \left( \tilde{Y}_1 \right) / \mathbb{R} \right)$, $(v_2, v_3) \in \prod_{i=2}^3 L^2 \left( Q; H^1_\# (Y_i) / \mathbb{R} \right)$ such that we have the following two-scale (denoted $\overset{2s}{\longrightarrow}$) convergences:

1. $u^\varepsilon(t, x) \overset{2s}{\longrightarrow} \chi_1(y) v_1(t, x, \tilde{y}) + \chi_2(y) u_2(t, x) + \chi_3(y) v_3(t, x, y)$
2. $(\chi_1^\varepsilon u^\varepsilon, \varepsilon \chi_1^\varepsilon \nabla^\varepsilon u^\varepsilon) \overset{2s}{\longrightarrow} (\chi_1(y) v_1(t, x, \tilde{y}), \chi_1(y) \nabla^\varepsilon v_1(t, x, \tilde{y}))$
3. $(\chi_2^\varepsilon u^\varepsilon, \chi_2^\varepsilon \nabla x u^\varepsilon) \overset{2s}{\longrightarrow} (\chi_2(y) u_2(t, x), \chi_2(y) \left[ \nabla x u_2(t, x) + \nabla y v_2(t, x, y) \right])$
4. $(\chi_3^\varepsilon u^\varepsilon, \varepsilon \chi_3^\varepsilon \nabla x u^\varepsilon) \overset{2s}{\longrightarrow} (\chi_3(y) v_3(t, x, y), \chi_3(y) \nabla y v_3(t, x, y))$
Moreover, there exists a unique function $w_3 \in L^2 \left(Q; H^1_\#(Y_3)\right)$ such that

\[
\begin{align*}
    v_3(t, x, y) &= u_2(t, x) + w_3(t, x, y) \text{ in } Y_3 \\
    w_3(t, x, y) &= v_1(t, x, \tilde{y}) - u_2(t, x) \text{ on } Y_{13} := \tilde{Y}_{13} \times I \\
    w_3(t, x, y) &= 0 \text{ on } Y_{23} := \tilde{Y}_{23} \times I
\end{align*}
\]

and $u^\varepsilon$ converges weakly in $L^2(Q)$ to the function

\[
U(t, x) = (1 - \theta_1) u_2(t, x) + \int_{\tilde{Y}_1} v_1(t, x, \tilde{y}) d\tilde{y} + \int_{Y_3} w_3(t, x, y) dy
\]
Homogenized Problems in the case $0 < \delta < \infty$

**Case 1 : $\gamma = 0$**

we shall introduce the following auxiliary functions : Let $w_k^2$, $k = 1, \ldots, N$, be the unique solutions of the cellular problems

\[
\begin{cases}
-\text{div}_y \left[ a_2(y) \left( \nabla_y w_k^2 + e_k \right) \right] = 0 \text{ in } Y_2 \\
a_2(y) \left( \nabla_y w_k^2 + e_k \right) \cdot n(y) = 0 \text{ on } Y_{23} \\
y \mapsto w_k^2(y), \quad a_2(y) \frac{\partial w_k^2}{\partial y} \cdot n(y) \bigg|_{\partial Y_2 \cap \partial Y} \quad Y - \text{periodic},
\end{cases}
\]

where $(e_1, e_2, \cdots, e_N)$ is the canonical basis of $\mathbb{R}^N$. Then, we define the homogenized matrix $\mathbb{A}^{2,\text{hom}}$ by its entries

\[
\mathbb{A}^{2,\text{hom}}_{kl} = \int_{Y_2} \left( \nabla_y w_k^2 + e_k \right) \left( \nabla_y w_l^2 + e_l \right) dy
\]
Homogenized Problems in the case $0 < \delta < \infty$

**Case 1: $\gamma = 0$**

Let $\eta_1, \eta_2$ and $\eta_3$ be the solutions of the following cellular problems

\[
\begin{aligned}
\text{(cell.3,1)} \\
\left\{ \begin{array}{l}
  c_3(y) \frac{\partial \eta_1}{\partial t}(t, y) - \text{div}_y [a_3(y) \nabla_y \eta_1] = \delta(t) \text{ in } Y_3 \\
  \eta_1(t, y) = 0 \text{ on } Y_{13}, \quad \eta_1(t, y) = 0 \text{ on } Y_{23} \\
  c_3(y)\eta_1(0, y) = 0 \text{ in } Y_3 \\
  y \mapsto \eta_1, \quad Y - \text{periodic}
\end{array} \right.
\end{aligned}
\]
Homogenized Problems in the case $0 < \delta < \infty$

- **Case 1 :** $\gamma = 0$

\[
\begin{cases}
  c_3(y) \frac{\partial \eta_2}{\partial t}(t, y) - \text{div}_y \left[ a_3(y) \nabla_y \eta_2 \right] = 0 \text{ in } Y_3 \\
  \eta_2(t, y) = 0 \text{ on } Y_{13}, \quad \eta_2(t, y) = 0 \text{ on } Y_{23} \\
  c_3(y) \eta_2(0, y) = c_3(y) \text{ in } Y_3 \\
  y \mapsto \eta_2, \quad Y - \text{periodic}
\end{cases}
\]
Homogenized Problems in the case $0 < \delta < \infty$

- **Case 1**: $\gamma = 0$

\[
\begin{aligned}
\begin{cases}
    c_3(y) \frac{\partial \eta_3}{\partial t}(t, y) - \text{div}_y [a_3(y) \nabla_y \eta_3] = 0 \text{ in } Y_3 \\
    \eta_3(t, y) = -\delta(t) \quad \text{on } Y_{13}, \quad \eta_3(t, y) = 0 \quad \text{on } Y_{23} \\
    c_3(y)\eta_3(0, y) = 1 \quad \text{in } Y_3 \\
    y \mapsto \eta_1, \quad Y - \text{periodic}
\end{cases}
\end{aligned}
\]

where $\delta(t)$ is the Dirac measure at $t = 0$. Then we define the following exchange coefficients

\[
K_{\alpha q}^\alpha(t) = \frac{1}{\delta} \int_{Y_{\alpha 3}} a_3(y) \frac{\partial \eta_q}{\partial n_3} dS(y), \quad \alpha = 1, 2, \quad q = 1, 2, 3.
\]
Homogenized Problems in the case $0 < \delta < \infty$

- **Case 1:** $\gamma = 0$

Theorem 1. 

\[(u_\alpha, v_2, w_3) \in L^2(0, T; H^1_{LB}(\Omega)) \times L^2(Q; H^1_\#(Y_2)/\mathbb{R}) \times L^2(Q; H^1_\#(Y_3))\]

are the unique solutions of the following homogenized coupled problems:

\[
\begin{align*}
\tilde{c}_1 \frac{\partial u_1}{\partial t} - a_1^{\text{hom}} \partial_{x_N}^2 u_1 + (u_1 - u_2)^* \left( \mathcal{K}_3^1 - \mathcal{K}_2^1 \right) &= \theta_1 f - f * \mathcal{K}_1^1 - \mathcal{K}_2^1 u_0 \text{ in } Q \\
\tilde{c}_2 \frac{\partial u_2}{\partial t} - \text{div}_x \left( \mathbb{A}^{2,\text{hom}} \nabla u_2 \right) + (u_1 - u_2)^* \left( \mathcal{K}_3^2 - \mathcal{K}_2^2 \right) &= \theta_2 f - f * \mathcal{K}_1^2 - \mathcal{K}_2^2 u_0 \text{ in } Q \\
\tilde{c}_\alpha u_\alpha(0, x) = \tilde{c}_\alpha u_0(x) &\text{ in } \Omega, \quad \tilde{c}_\alpha = \int_{Y_\alpha} c_\alpha(y) dy, \quad a_1^{\text{hom}} = \int_{\tilde{Y}_1} a_1(\tilde{y}) d\tilde{y} \\
u_\alpha(t, x) = 0 &\text{ on } (0, T) \times \partial\tilde{\Omega} \times [0, 1[ \\
w_3 = - \frac{\partial u_2}{\partial t} &\ast \eta_1 + \eta_1 \ast f + \eta_2 u_0 + \eta_3 \ast (u_1 - u_2), \quad v_2 = \frac{\partial u_2}{\partial x_k} w_k^2
\end{align*}
\]

where the prime in $\mathcal{K}_2^1, \mathcal{K}_2^2$ indicates derivation with respect to $t$. 
Homogenized Problems in the case $0 < \delta < \infty$

- **Case 2**: $0 < \gamma < \infty$

Theorem 2.

$$(u_2, v_1, v_2, w_3) \in L^2(0, T; H^1_{LB}(\Omega)) \times L^2(Q; H^1_\#(\tilde{Y}_1)/\mathbb{R}) \times L^2(Q; H^1_\#(Y_2)/\mathbb{R})$$

are the unique solutions of the following two-scale homogenized problems:

$$
\left\{ 
\begin{array}{l}
\tilde{c}_2 \frac{\partial u_2}{\partial t} - \text{div}_x \left( A^{2,\text{hom}} \nabla u_2 \right) + \frac{1}{\delta} \int_{Y_{23}} a_3(y) \frac{\partial w_3}{\partial n_3}(t, x, y) dS(y) = \\
\theta_2 f \text{ in } Q \\
\tilde{c}_2 u_2(0, x) = \tilde{c}_2 u_0(x) \text{ in } \Omega, \quad \tilde{c}_2 = \int_{Y_2} c_2(y) dy \\
u_2(t, x) = 0 \text{ on } (0, T) \times \partial \tilde{\Omega} \times [0, 1] \\
v_2 = \frac{\partial u_2}{\partial x_k} w_k
\end{array} \right.
$$
Homogenized Problems in the case $0 < \delta < \infty$

• **Case 2 : $0 < \gamma < \infty$**

Theorem 2.

\[
\begin{cases}
\langle c_1 \rangle_I \frac{\partial v_1}{\partial t} - \frac{1}{\gamma} \text{div}_\widetilde{\gamma} \left[ a_1(\widetilde{\gamma}) \nabla \widetilde{\gamma} v_1 \right] - a_1(\widetilde{\gamma}) \partial^2_{x_N} v_1 = f \text{ in } \widetilde{Y}_1 \\
 a_1(\widetilde{\gamma}) \frac{\partial v_1}{\partial n_3} = \frac{\gamma}{\delta} \left< a_3(y) \frac{\partial w_3}{\partial n_3} \right>_{I} \text{ on } (0, T) \times \widetilde{Y}_{13} \\
 \langle c_1 \rangle_I (\widetilde{\gamma}) v_1 = \langle c_1 \rangle_I (\widetilde{\gamma}) u_0 \quad \widetilde{\gamma} \in \widetilde{Y}_1
\end{cases}
\]

where $\langle . \rangle_I$ denotes the integration with respect to $y_N$ over $I$. 
Homogenized Problems in the case $0 < \delta < \infty$

**Case 2 : $0 < \gamma < \infty$**

Theorem 2.

\[
\begin{align*}
\text{(***)} & \quad \left\{ \begin{array}{l}
c_3(y) \left( \frac{\partial u_2}{\partial t}(t, x) + \frac{\partial w_3}{\partial t}(t, x, y) \right) - \frac{1}{\delta} \text{div}_y [a_3(y) \nabla_y w_3(t, x, y)] \\
= f(t, x) \text{ in } Y_3 \\
w_3(t, x, y) = v_1(t, x, \tilde{y}) - u_2(t, x) \quad \text{on } (0, T) \times Y_{13} \\
w_3(t, x, y) = 0 \quad \text{on } (0, T) \times Y_{23} \\
c_3(y)w_3(0, x, y) = c_3(y) \left( u_0(x) - u_2(0, x) \right) \quad y \in Y_3
\end{array} \right.
\]
Remark:

Contrary to the previous case, here the problems $(*)$, $(**)$ and $(***)$ involve both macroscopic and microscopic variables $(x, y)$, a unique macroscopic function $u_2$ and three microscopic functions $v_1, v_2, w_3$. The functions $v_1, w_3$ are ”strongly” couped via the boundary conditions

$$a_1(\tilde{y}) \frac{\partial v_1}{\partial n_3} = \frac{\gamma}{\delta} \left[ a_3(y) \frac{\partial w_3}{\partial n_3} \right]_I \text{ on } (0, T) \times \tilde{Y}_{13}$$

and $w_3(t, x, y) = v_1(t, x, \tilde{y}) - u_2(t, x) \text{ on } (0, T) \times Y_{13}$, therefore, the uncoupling between $v_1$ and $w_3$ could not be possible. As a consequence, we can not eliminate the microscopic variable. Moreover, it should be noted that the third term in the first equation in $(*)$, the second equation in $(**) \text{ and } (***)$ respectively might be interpreted as the combined effects of fiber coatings together with the highly anisotropy of the fibers in the overall behavior of the composite.
Remark:

These results should be compared with those of [Panasenko, 1991] and [Cherednichenko et al., 2006] as follows:

a) The structure of the homogenized operator in $(\ast)$, which can be viewed as the effective properties of the matrix, is the same as in [Panasenko, 1991] (see Theorem 6).

b) In the stationary case ($c_k = 0$ a.e. for $k \in \{1, 2, 3\}$), we obtain exactly the statement obtained by [Cherednichenko et al. 2006] (see Theorem 2.2).
Homogenized Problems in the case $\delta = 0$ and $\delta = \infty$

Theorem 3.

The case $\delta = 0$

- When $\gamma = 0$, the functions

$$(u, v_2) \in L^2(0, T; H^1_{LB}(\Omega)) \times L^2(Q; H^1_{#}(Y_2)/\mathbb{R})$$

are the unique solutions of the following homogenized problems:

$$
\begin{cases}
\tilde{c} \frac{\partial u}{\partial t} - a_{1}^{\text{hom}} \frac{\partial^2 u}{\partial x^2} - \text{div}_x \left( A^{2, \text{hom}} \nabla u \right) = f \text{ in } Q \\
\tilde{c} u(0, x) = \tilde{c} u_0(x) \text{ in } \Omega, \quad \tilde{c} = \int_Y c(y) dy, \quad a_{1}^{\text{hom}} = \int_{Y_1} a_1(\tilde{y}) d\tilde{y} \\
u(t, x) = 0 \text{ on } (0, T) \times \partial \tilde{\Omega} \times [0, 1[ \\
v_2 = \frac{\partial u}{\partial x_k} w_k
\end{cases}
$$
Homogenized Problems in the case $\delta = 0$ and $\delta = \infty$

Theorem 3.

The case $\delta = 0$

• When $0 < \gamma < \infty$, the functions

$$(u_2, v_1, v_2) \in L^2(0, T; H^1_{LB}(\Omega)) \times L^2(Q; H^1(\tilde{Y}_1)/\mathbb{R}) \times L^2(Q; H^1(Y_2)/\mathbb{R})$$

are the unique solutions of the following two-scale homogenized problems:

$$
\begin{aligned}
\tilde{c}_2 \frac{\partial u_2}{\partial t} - \text{div}_x \left( A^{2,\text{hom}} \nabla u_2 \right) &= \theta_2 f \text{ in } Q \\
\tilde{c}_2 u_2(0, x) &= \tilde{c}_2 u_0(x) \text{ in } \Omega, \quad \tilde{c}_2 = \int_{Y_2} c_2(y) dy \\
u_2(t, x) &= 0 \text{ on } (0, T) \times \partial \tilde{\Omega} \times [0, 1] \\
v_2 &= \frac{\partial u_2}{\partial x_k} w_k
\end{aligned}
$$
Homogenized Problems in the case $\delta = 0$ and $\delta = \infty$

Theorem 3.

The case $\delta = 0$
• When $0 < \gamma < \infty$, the functions

$$(u_2, v_1, v_2) \in L^2(0, T; H^1_{\text{LB}}(\Omega)) \times L^2(Q; H^1_{\#}(\tilde{Y}_1)/\mathbb{R}) \times L^2(Q; H^1_{\#}(Y_2)/\mathbb{R})$$

are the unique solutions of the following two-scale homogenized problems:

$$
\begin{align*}
\langle c_1 \rangle_I \frac{\partial v_1}{\partial t} &- \frac{1}{\gamma} \text{div}_{\tilde{y}} \left[ a_1(\tilde{y}) \nabla_{\tilde{y}} v_1 \right] - a_1(\tilde{y}) \partial_{x_n}^2 v_1 = f \text{ in } \tilde{Y}_1 \\
a_1(\tilde{y}) \frac{\partial v_1}{\partial n_1} & = 0 \text{ on } (0, T) \times \tilde{Y}_{13} \\
\langle c_1 \rangle_I (\tilde{y}) v_1 & = \langle c_1 \rangle_I (\tilde{y}) u_0 \quad \tilde{y} \in \tilde{Y}_1
\end{align*}
$$
Homogenized Problems in the case $\delta = 0$ and $\delta = \infty$

Theorem 3.
Homogenized Problems in the case $\delta = 0$ and $\delta = \infty$

Theorem 3.

The case $\delta = \infty$

- When $\gamma = 0$,

$$(u_\alpha, v_2, w_3) \in L^2(0, T; H_{LB}^1(\Omega)) \times L^2(Q; H_{\#}^1(Y_2)/\mathbb{R}) \times L^2(Q; H_{\#}^1(Y_3))$$

are the unique solutions of the following homogenized coupled problems:

$$\begin{cases}
\tilde{c}_1 \frac{\partial u_1}{\partial t} - a_1^{\text{hom}} \partial_{x_N} u_1(t, x) = \theta_1 f \text{ in } Q \\
\tilde{c}_2 \frac{\partial u_2}{\partial t} - \text{div}_x \left( A_2^{2,\text{hom}} \nabla u_2 \right) = \theta_2 f \text{ in } Q \\
\tilde{c}_\alpha u_\alpha(0, x) = \tilde{c}_\alpha u_0(x) \text{ in } \Omega, \quad \tilde{c}_\alpha = \int_{Y_\alpha} c_\alpha(y) dy, \quad a_1^{\text{hom}} = \int_{\tilde{Y}_1} a_1(\tilde{y}) d\tilde{y} \\
u_\alpha(t, x) = 0 \text{ on } (0, T) \times \partial\tilde{\Omega} \times [0, 1[ \\
w_3 = u_0, \text{ in } \Omega, \quad v_2 = \frac{\partial u_2}{\partial x_k} w_k^2
\end{cases}$$
Homogenized Problems in the case $\delta = 0$ and $\delta = \infty$

Theorem 3.

The case $\delta = \infty$

• When $0 < \gamma < \infty$,

$$(u_2, v_1, v_2, w_3) \in L^2(0, T; H^1_{LB}(\Omega)) \times L^2(Q; H^1_\#(\bar{Y}_1)/\mathbb{R}) \times L^2(Q; H^1_\#(Y_2)/\mathbb{R})$$

are the unique solutions of the following two-scale homogenized problems:

$$
\begin{cases}
\tilde{c}_2 \frac{\partial u_2}{\partial t} - \text{div}_x \left( \tilde{A}^{2,\text{hom}} \nabla u_2 \right) = \theta_2 f \text{ in } Q \\
\tilde{c}_2 u_2(0, x) = \tilde{c}_2 u_0(x) \text{ in } \Omega, \quad \tilde{c}_2 = \int_{Y_2} c_2(y) dy \\
w_3 = u_0, \quad y \in Y_3, \quad u_2(t, x) = 0 \text{ on } (0, T) \times \partial \bar{\Omega} \times [0, 1], \\
v_2 = \frac{\partial u_2}{\partial x_k} w_k.
\end{cases}
$$
Homogenized Problems in the case $\delta = 0$ and $\delta = \infty$

Theorem 3.

The case $\delta = \infty$

- When $0 < \gamma < \infty$,

$$(u_2, v_1, v_2, w_3) \in L^2(0, T; H^1_{LB}(\Omega)) \times L^2(Q; H^1(\tilde{Y}_1)/\mathbb{R}) \times L^2(Q; H^1(Y_2)/\mathbb{R})$$

are the unique solutions of the following two-scale homogenized problems:

\[
\begin{cases}
\langle c_1 \rangle_I \frac{\partial v_1}{\partial t} - \frac{1}{\gamma} \text{div}_{\tilde{Y}} \left[ a_1(\tilde{y}) \nabla_{\tilde{y}} v_1 \right] - a_1(\tilde{y}) \partial_{x_n}^2 v_1 = f \text{ in } \tilde{Y}_1 \\
a_1(\tilde{y}) \frac{\partial v_1}{\partial n_1} = 0 \text{ on } (0, T) \times \tilde{Y}_{13} \\
\langle c_1 \rangle_I (\tilde{y}) v_1 = \langle c_1 \rangle_I (\tilde{y}) u_0 \quad \tilde{y} \in \tilde{Y}_1
\end{cases}
\]