EXISTENCE AND ASYMPTOTIC STABILITY OF SOLUTIONS OF A PERTURBED QUADRATIC FRACTIONAL INTEGRAL EQUATION

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In this paper, we are interested in the existence and the asymptotic behaviour of solutions to the perturbed quadratic fractional integral equation

\begin{equation}
 x(t) = g(t, x(t)) + \frac{f(t, x(t))}{\Gamma(\beta)} \int_0^t v(t, s, x(s)) \frac{1}{(t-s)^{1-\beta}} \, ds,
\end{equation}

where $t \in \mathbb{R}_+ = [0, +\infty)$ and $0 < \beta < 1$. Throughout, $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $v : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are functions which satisfy special assumptions that will be given in detail later on.

The functions $f = f(t, x)$ and $g = g(t, x)$ involved in Eq. (0.1) generate superposition operators $F$ and $G$, respectively, defined by

\begin{equation}
 (Fx)(t) = f(t, x(t)) \text{ and } (Gx)(t) = g(t, x(t)),
\end{equation}

where $x = x(t)$ is an arbitrary function defined on $\mathbb{R}_+$.

We remark that:

- If $g(t, y) = p(t)$ in Eq. (0.1), then we have an equation studied by Banaś and O’Regan in [12].
- If $g(t, y) = a(t)$ and $v(t, s, x) = u(s, x)$ in Eq. (0.1), then we have an equation studied by Banaś and Rzepka in [11].
- If $g(t, y) = a(t)$, $f(t, y) = y$ and $v(t, s, x) = u(s, x)$ in Eq. (0.1), then we have an equation studied by Darwish in [23].

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Consider the limit case where $\beta = 1$. Let $g(t, x) = h(t)$, $f(t, x) = -x$, and $v(t, s, x) = k(t, s)x$. Then Eq. (0.1) takes the form
\begin{equation}
(0.3) \quad x(t) + x(t) \int_0^t k(t, s) x(s) \, ds = h(t), \ t \in [0, 1].
\end{equation}
Eq. (0.3) is the nonlinear particle transport equation when removal effects are dominant, where $t$ is the particle speed, the known term $h(t)$ is the intensity of the external source, and the unknown function $x(t)$ is related to the particle distribution function $y(t)$ by
\begin{equation}
x(t) = Q(t) y(t),
\end{equation}
where $Q$ is the positive macroscopic removal collision frequency of the host medium. Finally, the kernel $k(t, s)$ is given by
\begin{equation}
k(t, s) = \frac{1}{2tQ(t)Q(s)} \int_{|t-s|}^{t+s} v q(v) \, dv,
\end{equation}
where $q$ is the macroscopic removal collision frequency of the particles amongst themselves; see [15, 16, 17, 42]. On the other hand, Eq. (0.3) is a generalization of the Chandrasekhar $H$–equation in transport theory, in which $t$ ranges from 0 to 1, $h(t) = 1$, $x$ must be identified with the $H$–function, and
\begin{equation}
k(t, s) = -\frac{t\phi(s)}{t + s}
\end{equation}
for a nonnegative characteristic function $\phi$; see [22, 34, 36, 42].

Moreover, quadratic integral equations have numerous other useful applications in describing events and problems in the real world. For example, quadratic integral equations are often applicable in the kinetic theory of gases, in the theory of neutron transport, and in traffic theory; see [15, 16, 17, 29, 32, 34].

In the last 35 years or so, many authors have studied the existence of solutions for several classes of nonlinear quadratic integral equations with nonsingular kernels. For example, see the papers by Argyros [2], Banaš et al. [5, 7, 10], Banaš and Martinon [9], Benchohra and Darwish [14], Caballero et al. [18, 19, 20, 21], Darwish [25], Hu and Yan [33], Leggett [36], Liu and Kang [37], Stuart [41] and Spiga et al. [42].

More recently, following the appearance of the paper [23], there has been significant interest in the study of the existence of solutions for singular quadratic integral equations or fractional quadratic integral equations; see [11, 12, 13, 24, 26, 27, 28].

It is worth mentioning that up to now only the two papers by Banaš and D. O’Regan [12] and Darwish [26] have dealt with the study of quadratic integral equation of singular kernel in the space of real functions which are defined, continuous and
bounded on an \textit{unbounded interval}. The proofs in [12] and [26] depend on a suitable combination of the technique of measures of noncompactness and the Schauder fixed point principle.

The aim of this paper is to prove the existence of solutions to Eq. (0.1) in the space of real functions which are defined, continuous and bounded on an unbounded interval. Moreover, we will obtain some asymptotic characterizations of the solutions of Eq. (0.1). Our proof depends on a suitable combination of the technique of measures of noncompactness and the Darbo fixed point principle. Also, we give an example for indicating the natural realizations of our abstract theory presented in the paper.

\textbf{References}


